

M337

Complex analysis

Handbook

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Contents

Greek alphabet	4
Mathematical language	4
Set notation	5
Real numbers	6
Real functions	8
Book A	13
Unit A1 Complex numbers	13
Section 1 Complex numbers and their properties	13
Section 2 The complex plane	15
Section 3 Solving equations with complex numbers	17
Section 4 Sets of complex numbers	18
Section 5 Proving inequalities	20
Unit A2 Complex functions	21
Section 1 Complex functions and their properties	21
Section 2 Special types of complex function	22
Section 3 Images of grids	25
Section 4 Exponential, trigonometric and hyperbolic functions	25
Section 5 Logarithms and powers	28
Unit A3 Continuity	29
Section 1 Sequences	29
Section 2 Continuous functions	31
Section 3 Limits of functions	32
Section 4 Regions	33
Section 5 The Extreme Value Theorem	34
Unit A4 Differentiation	36
Section 1 Derivatives of complex functions	36
Section 2 The Cauchy–Riemann equations	38
Section 3 Rules for manipulating differentiable functions	39
Section 4 Smooth paths	40
Book B	41
Unit B1 Integration	41
Section 1 Integrating real functions	41
Section 2 Integrating complex functions	41
Section 3 Evaluating contour integrals	42
Section 4 Estimating contour integrals	43

Unit B2	Cauchy's Theorem	44
Section 1	Cauchy's Theorem	44
Section 2	Cauchy's Integral Formula	45
Section 3	Cauchy's Derivative Formulas	45
Section 4	Revision of contour integration	46
Section 5	Proof of Cauchy's Theorem	46
Unit B3	Taylor series	47
Section 1	Complex series	47
Section 2	Power series	49
Section 3	Taylor's Theorem	51
Section 4	Manipulating Taylor series	52
Section 5	The Uniqueness Theorem	54
Unit B4	Laurent series	55
Section 1	Singularities	55
Section 2	Laurent's Theorem	56
Section 3	Behaviour near a singularity	58
Section 4	Evaluating integrals using Laurent series	58
Book C		59
Unit C1	Residues	59
Section 1	Calculating residues	59
Section 2	The Residue Theorem	59
Section 3	Evaluating improper integrals	60
Section 4	Summing series	63
Section 5	Analytic continuation	64
Unit C2	Zeros and extrema	66
Section 1	Winding numbers	66
Section 2	Locating zeros of analytic functions	67
Section 3	Local behaviour of analytic functions	68
Section 4	Extreme values of analytic functions	69
Section 5	Uniform convergence	69
Section 6	Special functions	71
Unit C3	Conformal mappings	72
Section 1	Linear and reciprocal functions	72
Section 2	Möbius transformations	74
Section 3	Images of generalised circles	75
Section 4	Transforming regions	76

Book D	80
Unit D1 Fluid flows	80
Section 1 Setting up the model	80
Section 2 Complex potential functions	82
Section 3 The Joukowski functions	85
Section 4 Flow past an obstacle	86
Section 5 Flow past an aerofoil	87
Unit D2 The Mandelbrot set	88
Section 1 Iteration of analytic functions	88
Section 2 Iterating complex quadratics	89
Section 3 Graphical iteration	90
Section 4 The Mandelbrot set	91
Index	93

Greek alphabet

α	A	alpha	ι	I	iota	ρ	P	rho
β	B	beta	κ	K	kappa	σ	Σ	sigma
γ	Γ	gamma	λ	Λ	lambda	τ	T	tau
δ	Δ	delta	μ	M	mu	υ	Υ	upsilon
ε	E	epsilon	ν	N	nu	ϕ	Φ	phi
ζ	Z	zeta	ξ	Ξ	xi	χ	X	chi
η	H	eta	\omicron	O	omicron	ψ	Ψ	psi
θ	Θ	theta	π	Π	pi	ω	Ω	omega

Mathematical language

1. In mathematics we commonly use **implications** such as ‘if P , then Q ’, where P and Q are statements which can either be true or false. The statement P is the **hypothesis** and the statement Q is the **conclusion**.

An implication may be true or it may be false. For example, the following implication is true:

if x is positive, then $x + 1$ is positive.

In contrast, the following implication is false:

if x is positive, then $x - 1$ is positive.

One way to prove that an implication is false is by giving a **counterexample** to the implication. For example, the implication ‘if x is positive, then $x - 1$ is positive’ is false because $x = 1$ is a counterexample to the implication, since x is positive but $x - 1 = 0$ is not positive.

There are various equivalent ways of stating an implication. For example:

- if x is positive, then $x + 1$ is positive
 - if $x > 0$, then $x + 1 > 0$
 - $x > 0 \implies x + 1 > 0$
 - for all $x > 0$, we have $x + 1 > 0$
 - $x + 1 > 0$, for all $x > 0$
 - $x + 1 > 0$, whenever $x > 0$
 - for $x + 1$ to be positive, it is sufficient that x be positive.
2. The **converse** of an implication is obtained by exchanging the hypothesis and the conclusion. The converse of a true implication is not necessarily true. For example, the converse of the (true) implication

if $x > 0$, then $x + 1 > 0$

is

if $x + 1 > 0$, then $x > 0$,

which is false (for example, if $x = 0$ then $x + 1 > 0$ but $x \leq 0$).

3. An **equivalence** consists of an implication ‘if P , then Q ’ *and* its converse ‘if Q , then P ’. The equivalence is true if both these implications are true. For example, the following equivalence is true:

$$x > 0 \text{ is equivalent to } 2x > 0.$$

It could alternatively be stated as follows:

- $x > 0 \iff 2x > 0$
 - $x > 0$ if and only if $2x > 0$
 - $x > 0$ is necessary and sufficient for $2x > 0$.
4. There are three ways of proving implications:
- **direct proof**: begin by assuming that the hypothesis is true and then argue directly to show that the conclusion is true
 - **proof by contraposition**: begin by assuming that the conclusion is false and then argue directly to show that this assumption implies that the hypothesis is false
 - **proof by contradiction**: begin by assuming that the hypothesis is true *and* that the conclusion is false, and then argue from both to obtain a contradiction.

It is preferable to use one of the first two types of proof, where possible, since they establish a direct link between hypothesis and conclusion.

However, it is often convenient and sometimes essential to use a proof by contradiction.

Set notation

Notation	Meaning
$\{x, y, \dots, z\}$	The set of elements listed in $\{\dots\}$
$\{x : \dots\}$	The set of all x such that \dots
$x \in A$	x belongs to A
$x \notin A$	x does not belong to A
$A \subseteq B$	A is a subset of B : each element of A belongs to B
$A = B$	A is equal to B : $A \subseteq B$ and $B \subseteq A$
$A \subset B$	A is a proper subset of B : $A \subseteq B$ but $A \neq B$
$A \cup B$	A union B : the set of all elements that belong to A or B (or both)
$A \cap B$	A intersection B : the set of all elements that belong to both A and B
$A - B$	A minus B : the set of all elements of A that do not belong to B
\emptyset	The empty set

Real numbers

1. A **real number** is a number that can be represented by a decimal of the form

$$\pm a_0.a_1a_2a_3\ldots,$$

where a_0 is a non-negative integer and a_1, a_2, a_3, \ldots are digits.

Rational numbers (ratios of integers) are represented by recurring decimals and **irrational numbers** are represented by non-recurring decimals. Real numbers are often represented by points on a line, called the **real line**.

2. **Some important sets of real numbers**

Symbol	Set
\mathbb{N}	The set of all natural numbers: $\{1, 2, 3, \ldots\}$
\mathbb{Z}	The set of all integers: $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
\mathbb{Q}	The set of all rational numbers (numbers of the form p/q , where $p, q \in \mathbb{Z}$, $q \neq 0$)
\mathbb{R}	The set of all real numbers
(a, b)	$\{x : a < x < b\}$, open interval
$[a, b]$	$\{x : a \leq x \leq b\}$, closed interval
$(a, b]$	$\{x : a < x \leq b\}$, half-open interval
(a, ∞)	$\{x : x > a\}$, open interval
$[a, \infty)$	$\{x : x \geq a\}$, closed interval
$(-\infty, b)$	$\{x : x < b\}$, open interval
$(-\infty, b]$	$\{x : x \leq b\}$, closed interval

3. **Upper and lower bounds**

Suppose that A is a non-empty subset of \mathbb{R} . Then A is **bounded above** if there is a real number M such that

$$x \leq M, \quad \text{for all } x \in A.$$

The number M is called an **upper bound** of A . Clearly, any number bigger than M is also an upper bound of A . A **lower bound** of A is defined similarly.

Among all upper bounds of A , the smallest (which always exists if A is bounded above) is called the **least upper bound** of A , or the **supremum** of A , written $\sup A$. The **greatest lower bound**, or the **infimum**, of A , written $\inf A$, is defined similarly.

If A has infinitely many elements and is bounded above, then $\sup A$ may or may not belong to A . Similarly for $\inf A$ when A has a lower bound. In contrast, if A has finitely many elements, then $\sup A$ and $\inf A$ are the largest and the smallest elements of A , respectively.

If $\sup A$ belongs to A , then we may use the alternative notation $\max A$ for $\sup A$. Similarly, if $\inf A$ belongs to A , then we may denote it by $\min A$.

4. Inequalities

Rules for rearranging inequalities

For all $a, b, c \in \mathbb{R}$, the following rules apply.

Rule 1 $a < b \iff b - a > 0$.

Rule 2 $a < b \iff a + c < b + c$.

Rule 3 If $c > 0$, then $a < b \iff ac < bc$.

If $c < 0$, then $a < b \iff ac > bc$.

Rule 4 If $a, b > 0$, then $a < b \iff \frac{1}{a} > \frac{1}{b}$.

Rule 5 If $a, b \geq 0$ and $p > 0$, then $a < b \iff a^p < b^p$.

Rules for deducing new inequalities from given ones

(a) **Transitive Rule** For all a, b, c in \mathbb{R} ,

$$a < b \text{ and } b < c \implies a < c.$$

(b) **Sum Rule** For all a, b, c, d in \mathbb{R} ,

$$a < b \text{ and } c < d \implies a + c < b + d.$$

(c) **Product Rule** For all a, b, c, d in \mathbb{R} with $a, c \geq 0$,

$$a < b \text{ and } c < d \implies ac < bd.$$

5. Modulus

If $x \in \mathbb{R}$, then the **modulus**, or **absolute value**, of x is

$$|x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Thus $|x|$ is the distance on the real line from the origin to x , so

- $|x| < a \iff -a < x < a$
- $|x| > a \iff x > a \text{ or } x < -a$
- the distance on the real line from a to b is $|b - a|$.

6. The Principle of Mathematical Induction

Suppose that $P(n)$, $n = 1, 2, \dots$, is a sequence of propositions such that

- $P(1)$ is true, and
- if $P(k)$ is true, then $P(k + 1)$ is also true.

Then $P(n)$ is true, for $n = 1, 2, \dots$.

Real functions

1. A **real function** f is defined by specifying
- two subsets A and B of \mathbb{R}
 - a rule that associates with each $x \in A$ a unique $y \in B$.

We write

$$f: A \longrightarrow B \quad \text{and} \quad y = f(x).$$

The sets A and B are called the **domain** and the **codomain** of f , respectively. The number y is called the **image of x under f** , or the **value of f at x** , and we say that **f maps x to y** . The **image set** of the function f is

$$f(A) = \{f(x) : x \in A\}.$$

2. **Standard functions**

Type	Rule	Domain
Polynomial	$a_0 + a_1x + \cdots + a_nx^n$	\mathbb{R}
Rational	$p(x)/q(x)$, p and q are polynomial functions (q not the zero function)	$\mathbb{R} - \{x : q(x) = 0\}$
Trigonometric	$\sin x$	\mathbb{R}
	$\cos x$	\mathbb{R}
	$\tan x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
Exponential	e^x (also written $\exp x$)	\mathbb{R}
	a^x , where $a > 0$	\mathbb{R}
Natural log	$\log x$	$(0, \infty)$
Hyperbolic	$\sinh x$	\mathbb{R}
	$\cosh x$	\mathbb{R}
	$\tanh x$	\mathbb{R}

3. **Properties of real exponentials and logarithms**

- (a) Definition of e^x :

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

- (b) Definition of $\log x$ (where $x > 0$, $y \in \mathbb{R}$):

$$x = e^y \iff y = \log x.$$

- (c) Definition of a^x (where $a > 0$, $x \in \mathbb{R}$):

$$a^x = \exp(x \log a).$$

- (d) Index laws (where $a > 0$, $x, y \in \mathbb{R}$, $m, n \in \mathbb{N}$):

$$a^{x+y} = a^x a^y, \quad (a^x)^y = a^{xy}, \quad \sqrt[n]{a^m} = a^{m/n}, \quad a^{-x} = 1/a^x.$$

- (e) Logarithmic identities (where $x, y > 0$):

$$\log xy = \log x + \log y, \quad \log(1/x) = -\log x.$$

4. Trigonometric and hyperbolic identities

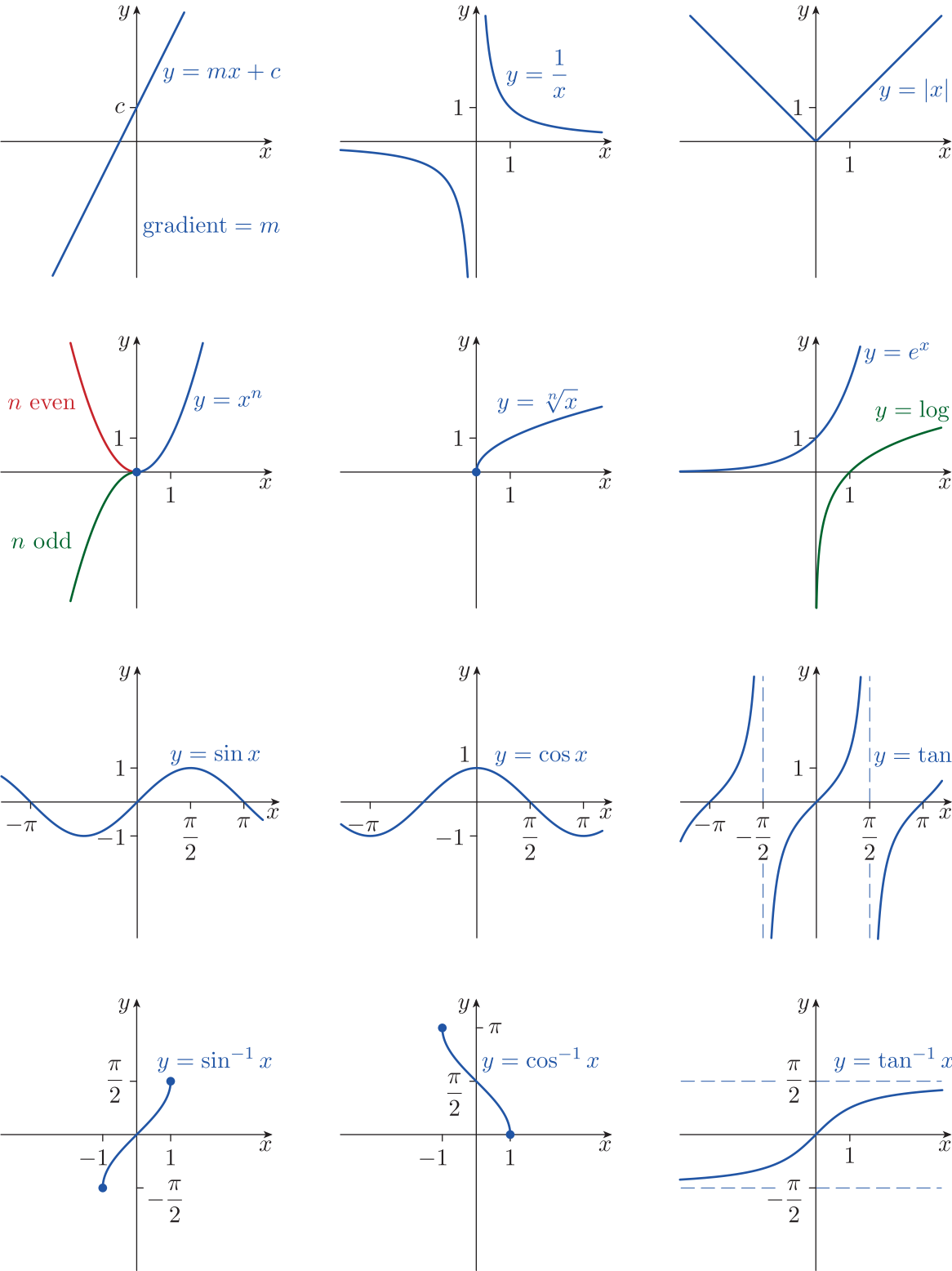
These identities are valid when z , z_1 and z_2 are real numbers *and* when z , z_1 and z_2 are complex numbers. Each identity holds on the largest set of values for which both sides of the identity are defined.

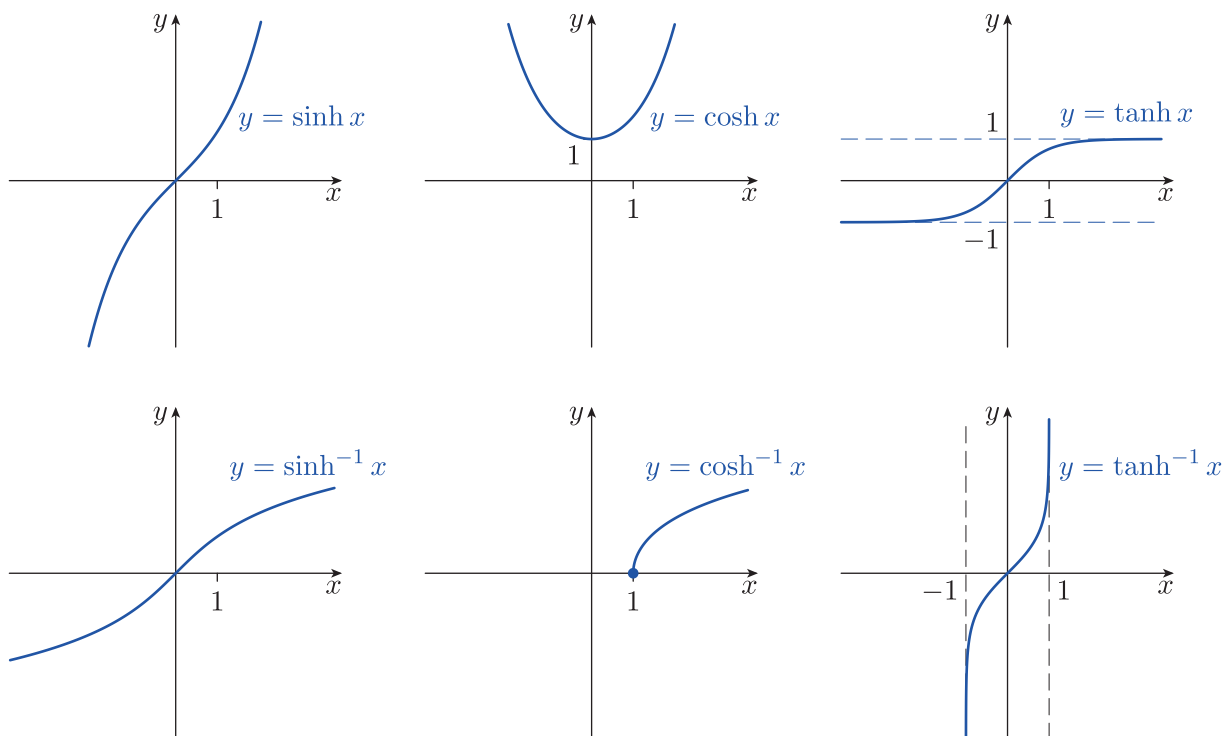
Trigonometric	Hyperbolic
$\cos^2 z + \sin^2 z = 1$ $\sec^2 z = 1 + \tan^2 z$ $\operatorname{cosec}^2 z = \cot^2 z + 1$	$\cosh^2 z - \sinh^2 z = 1$ $\operatorname{sech}^2 z = 1 - \tanh^2 z$ $\operatorname{cosech}^2 z = \coth^2 z - 1$
$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$ $\cos(z_1 \pm z_2) = \cos z_1 \cos z_2 \mp \sin z_1 \sin z_2$ $\tan(z_1 \pm z_2) = \frac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$	$\sinh(z_1 \pm z_2) = \sinh z_1 \cosh z_2 \pm \cosh z_1 \sinh z_2$ $\cosh(z_1 \pm z_2) = \cosh z_1 \cosh z_2 \pm \sinh z_1 \sinh z_2$ $\tanh(z_1 \pm z_2) = \frac{\tanh z_1 \pm \tanh z_2}{1 \pm \tanh z_1 \tanh z_2}$
$\sin 2z = 2 \sin z \cos z$ $\cos 2z = \cos^2 z - \sin^2 z$ $\quad = 2 \cos^2 z - 1$ $\quad = 1 - 2 \sin^2 z$ $\tan 2z = \frac{2 \tan z}{1 - \tan^2 z}$	$\sinh 2z = 2 \sinh z \cosh z$ $\cosh 2z = \cosh^2 z + \sinh^2 z$ $\quad = 2 \cosh^2 z - 1$ $\quad = 1 + 2 \sinh^2 z$ $\tanh 2z = \frac{2 \tanh z}{1 + \tanh^2 z}$
$\sin(-z) = -\sin z$ $\cos(-z) = \cos z$ $\tan(-z) = -\tan z$	$\sinh(-z) = -\sinh z$ $\cosh(-z) = \cosh z$ $\tanh(-z) = -\tanh z$
$\sin(z + 2\pi) = \sin z$ $\cos(z + 2\pi) = \cos z$ $\tan(z + \pi) = \tan z$	$\sinh(z + 2\pi i) = \sinh z$ $\cosh(z + 2\pi i) = \cosh z$ $\tanh(z + \pi i) = \tanh z$

5. Commonly used trigonometric values

θ in radians	θ in degrees	$\sin \theta$	$\cos \theta$	$\tan \theta$
0	0°	0	1	0
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{2}$	90°	1	0	undefined
$\frac{2\pi}{3}$	120°	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$\frac{3\pi}{4}$	135°	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
$\frac{5\pi}{6}$	150°	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
π	180°	0	-1	0

6. Graphs of standard functions





7. Standard derivatives

$f(x)$	$f'(x)$	Domain of f'
$k, k \in \mathbb{R}$	0	\mathbb{R}
$x^n, n = 1, 2, \dots$	nx^{n-1}	\mathbb{R}
$x^n, n = -1, -2, \dots$	nx^{n-1}	$\mathbb{R} - \{0\}$
$x^\alpha, \alpha \in \mathbb{R} - \mathbb{Z}$	$\alpha x^{\alpha-1}$	$(0, \infty)$
e^x	e^x	\mathbb{R}
$a^x, a > 0$	$a^x \log a$	\mathbb{R}
$\log x$	$1/x$	$(0, \infty)$
$\sin x$	$\cos x$	\mathbb{R}
$\cos x$	$-\sin x$	\mathbb{R}
$\tan x$	$\sec^2 x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sec x$	$\sec x \tan x$	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\cot x$	$-\operatorname{cosec}^2 x$	$\mathbb{R} - \{n\pi : n \in \mathbb{Z}\}$
$\sin^{-1} x$	$1/\sqrt{1-x^2}$	$(-1, 1)$
$\cos^{-1} x$	$-1/\sqrt{1-x^2}$	$(-1, 1)$
$\tan^{-1} x$	$1/(1+x^2)$	\mathbb{R}
$\sinh x$	$\cosh x$	\mathbb{R}
$\cosh x$	$\sinh x$	\mathbb{R}
$\tanh x$	$\operatorname{sech}^2 x$	\mathbb{R}
$\sinh^{-1} x$	$1/\sqrt{1+x^2}$	\mathbb{R}
$\cosh^{-1} x$	$1/\sqrt{x^2-1}$	$(1, \infty)$
$\tanh^{-1} x$	$1/(1-x^2)$	$(-1, 1)$

8. Standard primitives

$f(x)$	$F(x) \quad (F' = f)$	Domain of F
$x^n, \ n = 0, 1, 2, \dots$	$x^{n+1}/(n+1)$	\mathbb{R}
x^{-1}	$\log x $	$\mathbb{R} - \{0\}$
$x^n, \ n = -2, -3, \dots$	$x^{n+1}/(n+1)$	$\mathbb{R} - \{0\}$
$x^\alpha, \ \alpha \in \mathbb{R} - \mathbb{Z}$	$x^{\alpha+1}/(\alpha+1)$	$(0, \infty)$
e^x	e^x	\mathbb{R}
$a^x, \ a > 0$	$a^x/\log a$	\mathbb{R}
$\log x$	$x \log x - x$	$(0, \infty)$
$\sin x$	$-\cos x$	\mathbb{R}
$\cos x$	$\sin x$	\mathbb{R}
$\tan x$	$\log \sec x $	$\mathbb{R} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$1/\sqrt{1-x^2}$	$\begin{cases} \sin^{-1} x \\ -\cos^{-1} x \end{cases}$	$(-1, 1)$ $(-1, 1)$
$1/(1+x^2)$	$\tan^{-1} x$	\mathbb{R}
$\sinh x$	$\cosh x$	\mathbb{R}
$\cosh x$	$\sinh x$	\mathbb{R}
$\tanh x$	$\log(\cosh x)$	\mathbb{R}
$1/\sqrt{1+x^2}$	$\sinh^{-1} x$	\mathbb{R}
$1/\sqrt{x^2-1}$	$\cosh^{-1} x$	$(1, \infty)$
$1/(1-x^2)$	$\tanh^{-1} x$	$(-1, 1)$

Unit A1 Complex numbers

Section 1 Complex numbers and their properties

1. A **complex number** z is an expression of the form $x + iy$, where x and y are real numbers and i is a symbol with the property that $i^2 = -1$. We write

$$z = x + iy \quad \text{or, equivalently,} \quad z = x + yi,$$

and say that z is expressed in **Cartesian form**. The real number x is the **real part** of z (written $x = \operatorname{Re} z$) and the real number y is the **imaginary part** of z (written $y = \operatorname{Im} z$).

Two complex numbers are **equal** if their real parts are equal *and* their imaginary parts are equal.

The set of all complex numbers is denoted by \mathbb{C} .

2. The binary operations of **addition**, **subtraction** and **multiplication** of complex numbers are denoted by the same symbols as for real numbers and are performed by the usual procedure – that is, treating complex numbers as real expressions together with an algebraic symbol i with the property that $i^2 = -1$.
3. The **negative** $-z$ of a complex number $z = x + iy$ is

$$-z = (-x) + i(-y),$$

usually written $-z = -x - iy$.

4. The **reciprocal** $1/z$ of a non-zero complex number $z = x + iy$ is

$$\frac{1}{z} = \frac{x - iy}{x^2 + y^2}.$$

The alternative notation z^{-1} is also used for the reciprocal.

The **quotient** z_1/z_2 of a complex number z_1 by a non-zero complex number z_2 is

$$\frac{z_1}{z_2} = z_1 \left(\frac{1}{z_2} \right).$$

5. **Strategy for obtaining a quotient** To obtain the quotient

$$\frac{x_1 + iy_1}{x_2 + iy_2}, \quad \text{where } y_2 \neq 0,$$

in Cartesian form, multiply both numerator and denominator by $x_2 - iy_2$, so that the denominator becomes real.

6. The **complex conjugate** \bar{z} of a complex number $z = x + iy$ is

$$\bar{z} = x - iy.$$

7. $\operatorname{Re} \bar{z} = \operatorname{Re} z$ and $\operatorname{Im} \bar{z} = -\operatorname{Im} z$.

8. Some formulas using i

- (a) $\operatorname{Re} i = 0$ and $\operatorname{Im} i = 1$.
- (b) $i^2 = -1$, $i^3 = -i$, $i^4 = i^0 = 1$.
- (c) $1/i = -i$.
- (d) $\bar{i} = -i$.

9. Properties of the complex conjugate

- (a) If z is a complex number, then
 - (i) $z + \bar{z} = 2 \operatorname{Re} z$
 - (ii) $z - \bar{z} = 2i \operatorname{Im} z$
 - (iii) $\overline{\bar{z}} = z$.
- (b) If z_1 and z_2 are complex numbers, then
 - (i) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
 - (ii) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$
 - (iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
 - (iv) $\overline{z_1 / z_2} = \bar{z}_1 / \bar{z}_2$, where $z_2 \neq 0$.

10. Arithmetic in \mathbb{C}

Property	Addition	Multiplication
Closure	A1 For all z_1, z_2 in \mathbb{C} , $z_1 + z_2 \in \mathbb{C}$.	M1 For all z_1, z_2 in \mathbb{C} , $z_1 z_2 \in \mathbb{C}$.
Identity	A2 For all z in \mathbb{C} , $z + 0 = 0 + z = z$.	M2 For all z in \mathbb{C} , $z1 = 1z = z$.
Inverse	A3 For all z in \mathbb{C} , $z + (-z) = (-z) + z = 0$.	M3 For all non-zero z in \mathbb{C} , $zz^{-1} = z^{-1}z = 1$.
Associative	A4 For all z_1, z_2, z_3 in \mathbb{C} , $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.	M4 For all z_1, z_2, z_3 in \mathbb{C} , $(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
Commutative	A5 For all z_1, z_2 in \mathbb{C} , $z_1 + z_2 = z_2 + z_1$.	M5 For all z_1, z_2 in \mathbb{C} , $z_1 z_2 = z_2 z_1$.
Distributive	D For all z_1, z_2, z_3 in \mathbb{C} , $z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$.	

11. Let n and k be integers with $n \geq k \geq 0$. Then the **binomial coefficient** $\binom{n}{k}$ is given by

$$\binom{n}{k} = \frac{n!}{k! (n-k)!} = \frac{n(n-1) \cdots (n-k+1)}{k!},$$

where $0! = 1$.

12. Binomial Theorem

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\begin{aligned}(1+z)^n &= \sum_{k=0}^n \binom{n}{k} z^k \\ &= 1 + nz + \frac{n(n-1)}{2!} z^2 + \cdots + z^n.\end{aligned}$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$\begin{aligned}(z_1 + z_2)^n &= \sum_{k=0}^n \binom{n}{k} z_1^{n-k} z_2^k \\ &= z_1^n + nz_1^{n-1}z_2 + \frac{n(n-1)}{2!} z_1^{n-2} z_2^2 + \cdots + z_2^n.\end{aligned}$$

13. Geometric Series Identity

(a) If $z \in \mathbb{C}$ and $n \in \mathbb{N}$, then

$$1 - z^n = (1 - z)(1 + z + z^2 + \cdots + z^{n-1})$$

and

$$1 + z + z^2 + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}, \quad \text{for } z \neq 1.$$

(b) If $z_1, z_2 \in \mathbb{C}$ and $n \in \mathbb{N}$, then

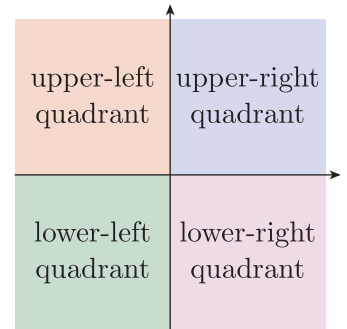
$$z_1^n - z_2^n = (z_1 - z_2)(z_1^{n-1} + z_1^{n-2}z_2 + z_1^{n-3}z_2^2 + \cdots + z_2^{n-1}).$$

Section 2 The complex plane

1. The **complex plane** or **z -plane** is a Cartesian plane used to represent the set of all complex numbers in which the complex number $z = x + iy$ is represented by the point (x, y) .

The horizontal axis of the complex plane is called the **real axis** and the vertical axis is called the **imaginary axis**.

The four infinite regions of the complex plane separated off by (and not including) the axes are called **quadrants**.



2. The **modulus**, or **absolute value**, of a complex number $z = x + iy$ is the distance from 0 to z ; it is denoted by $|z|$. Thus

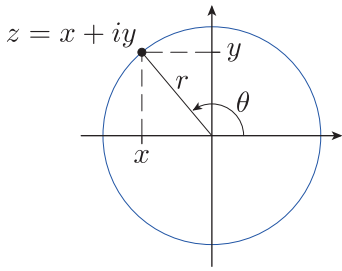
$$|z| = |x + iy| = \sqrt{x^2 + y^2}.$$

3. $|z_1 - z_2|$ is the distance from z_2 to z_1 .

$|z_1 + z_2|$ is the distance from $-z_2$ to z_1 .

4. Properties of the modulus

- (a) $|z| \geq 0$, with equality if and only if $z = 0$.
 (b) $|\bar{z}| = |z|$ and $|-z| = |z|$.
 (c) $|z|^2 = z\bar{z}$.
 (d) $|z_1 - z_2| = |z_2 - z_1|$.
 (e) $|z_1 z_2| = |z_1| |z_2|$ and $|z_1 / z_2| = |z_1| / |z_2|$, for $z_2 \neq 0$.



5. An **argument** of a non-zero complex number $z = x + iy$ with $|z| = r$ is an angle θ (measured in radians) such that

$$\cos \theta = \frac{x}{r} \quad \text{and} \quad \sin \theta = \frac{y}{r}.$$

6. No argument is assigned to the number 0.

Each non-zero complex number has infinitely many arguments, all differing by integer multiples of 2π .

7. The ordered pair (r, θ) , where r is the modulus of a non-zero complex number z and θ is an argument of z , is called the **polar coordinates** of z . The expression

$$z = r(\cos \theta + i \sin \theta)$$

is said to be a representation of z in **polar form**.

8. The **principal argument** of a non-zero complex number z is the unique argument θ of z satisfying $-\pi < \theta \leq \pi$; it is denoted by

$$\theta = \text{Arg } z.$$

(For $\text{Arg}_\phi z$, where $\phi \in \mathbb{R}$, see item 1 in Section 5 of Unit C1.)

9. **Strategy for determining principal arguments** To determine the principal argument θ of a non-zero complex number $z = x + iy$, apply the relevant case below.

Case 1 If z lies on one of the axes, then θ is evident.

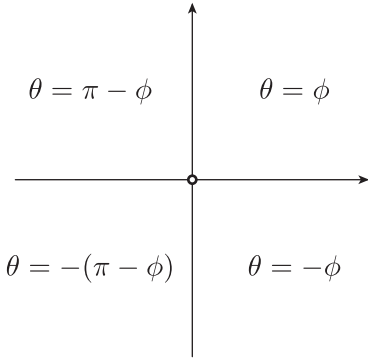
Case 2 If z does not lie on one of the axes, then carry out the following two steps.

- (1) Decide in which quadrant z lies (by plotting z if necessary), and then calculate the acute angle

$$\phi = \tan^{-1}(|y|/|x|)$$

in radians.

- (2) Obtain θ in terms of ϕ by using the appropriate formula in the figure.



10. Two non-zero complex numbers z_1 and z_2 are equal if and only if $|z_1| = |z_2|$ and $\text{Arg } z_1 = \text{Arg } z_2$.
11. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2),$$

then

$$z_1 z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)).$$

The geometric effect on z_1 of multiplying it by z_2 is to scale z_1 by the factor $|z_2|$ and rotate it about 0 through the angle $\text{Arg } z_2$. (This rotation is anticlockwise if $\text{Arg } z_2 > 0$ and clockwise if $\text{Arg } z_2 < 0$.)

12. If z_1 and z_2 are (non-zero) complex numbers, then

$$\text{Arg}(z_1 z_2) = \text{Arg } z_1 + \text{Arg } z_2 + 2n\pi,$$

where n is $-1, 0$ or 1 , depending on whether $\text{Arg } z_1 + \text{Arg } z_2$ is greater than π , lies in the interval $(-\pi, \pi]$, or is less than or equal to $-\pi$.

13. If z_k is non-zero with

$$z_k = r_k(\cos \theta_k + i \sin \theta_k), \quad \text{for } k = 1, 2, \dots, n,$$

then

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n (\cos(\theta_1 + \theta_2 + \cdots + \theta_n) + i \sin(\theta_1 + \theta_2 + \cdots + \theta_n)).$$

14. If z_1 and z_2 are non-zero with

$$z_1 = r_1(\cos \theta_1 + i \sin \theta_1) \quad \text{and} \quad z_2 = r_2(\cos \theta_2 + i \sin \theta_2),$$

then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)).$$

The geometric effect on z_1 of dividing it by z_2 is to scale z_1 by the factor $1/|z_2|$ and rotate it about 0 through the angle $-\text{Arg } z_2$. (This rotation is clockwise if $\text{Arg } z_2 > 0$ and anticlockwise if $\text{Arg } z_2 < 0$.)

15. If z is non-zero with $z = r(\cos \theta + i \sin \theta)$, then

$$z^{-1} = \frac{1}{r} (\cos(-\theta) + i \sin(-\theta)) = \frac{1}{r} (\cos \theta - i \sin \theta).$$

16. If z is non-zero and $-\pi < \text{Arg } z < \pi$, then

$$\text{Arg } \bar{z} = \text{Arg } z^{-1} = -\text{Arg } z.$$

17. **De Moivre's Theorem** If n is an integer and θ is a real number, then

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

18. **Some properties of i**

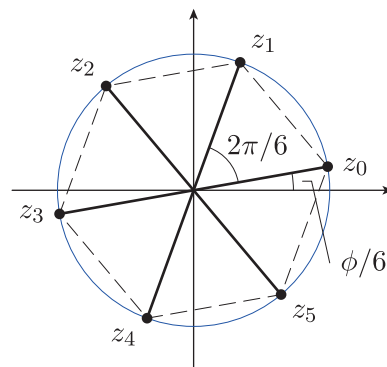
- (a) $|i| = 1$.
- (b) $\text{Arg } i = \pi/2$.
- (c) The arguments of i are $\pi/2 + 2n\pi$, for $n \in \mathbb{Z}$.
- (d) $i = \cos \pi/2 + i \sin \pi/2$ is a representation of i in polar form.

Section 3 Solving equations with complex numbers

- Let w be a non-zero complex number and let $n \geq 2$. Each solution of $z^n = w$ is called an **n th root** of w ; if $n = 2$ it is called a **square root**.
- Theorem** Let $w = \rho(\cos \phi + i \sin \phi)$ be a non-zero complex number in polar form. Then w has exactly n n th roots, given by

$$z_k = \rho^{1/n} \left(\cos \left(\frac{\phi}{n} + k \frac{2\pi}{n} \right) + i \sin \left(\frac{\phi}{n} + k \frac{2\pi}{n} \right) \right),$$

where $k = 0, 1, \dots, n-1$. These roots form the vertices of an n -sided regular polygon inscribed in the circle of radius $\rho^{1/n}$ centred at 0.



3. Let $w = \rho(\cos \phi + i \sin \phi)$ be a non-zero complex number in polar form, where ϕ is the principal argument of w . Then

$$z_0 = \rho^{1/n} \left(\cos \frac{\phi}{n} + i \sin \frac{\phi}{n} \right)$$

is called the **principal n th root** of w , denoted by $\sqrt[n]{w}$ or $w^{1/n}$.

By definition, $0^{1/n} = 0$.

4. The number 1 has exactly n n th roots, given by

$$z_k = \cos \left(\frac{2\pi k}{n} \right) + i \sin \left(\frac{2\pi k}{n} \right), \quad k = 0, 1, \dots, n-1.$$

These are called the **n th roots of unity**.

5. **Strategy for finding n th roots** To find the n n th roots z_0, z_1, \dots, z_{n-1} of a non-zero complex number w :

- (1) Express w in polar form, with modulus ρ and argument ϕ .
- (2) Substitute the values of ρ and ϕ in the formula

$$z_k = \rho^{1/n} \left(\cos \left(\frac{\phi}{n} + k \frac{2\pi}{n} \right) + i \sin \left(\frac{\phi}{n} + k \frac{2\pi}{n} \right) \right),$$

where $k = 0, 1, \dots, n-1$.

- (3) Convert the roots to Cartesian form, if required.

6. The solutions of the quadratic equation $az^2 + bz + c = 0$, where a, b, c are complex numbers and $a \neq 0$, are given by the quadratic formula

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Section 4 Sets of complex numbers

1. The inequalities $z_1 < z_2$ and $z_1 \leq z_2$ have no meaning unless both z_1 and z_2 are real.
2. The following subsets of \mathbb{C} are commonly used.
 - (a) A **half-plane** is the set of points lying to one side of a straight line, possibly including the line itself.

An **open half-plane** is a half-plane of the form

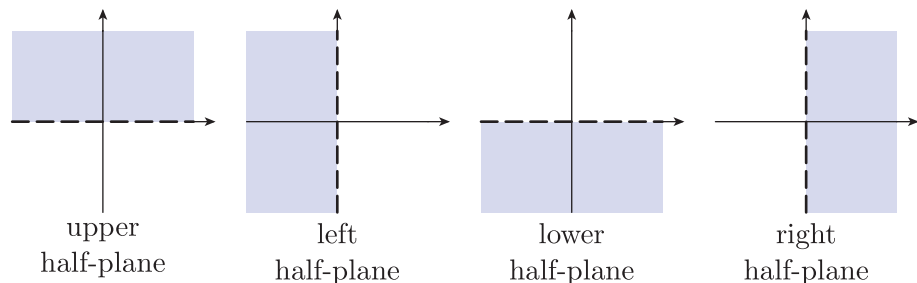
$$\{z : a \operatorname{Re} z + b \operatorname{Im} z > c\},$$

and a **closed half-plane** is a half-plane of the form

$$\{z : a \operatorname{Re} z + b \operatorname{Im} z \geq c\},$$

where $a, b, c \in \mathbb{R}$ and a, b are not both zero.

Four particularly useful open half-planes are shown in the figure.



- (b) The circle with centre $\alpha \in \mathbb{C}$ and radius $r > 0$ can be written as $\{z : |z - \alpha| = r\}$. The circle $\{z : |z| = 1\}$ with centre 0 and radius 1 is called the **unit circle**.
- (c) A **disc** is the set of points lying inside a circle, possibly including the circle itself.

An **open disc** is a disc of the form

$$\{z : |z - \alpha| < r\},$$

and a **closed disc** is a disc of the form

$$\{z : |z - \alpha| \leq r\},$$

where $\alpha \in \mathbb{C}$ and $r > 0$.

- (d) An **annulus** is the set of points lying between two concentric circles, possibly including one or both of the boundary circles.

An **open annulus** is an annulus of the form

$$\{z : r_1 < |z - \alpha| < r_2\},$$

and a **closed annulus** is an annulus of the form

$$\{z : r_1 \leq |z - \alpha| \leq r_2\},$$

where $\alpha \in \mathbb{C}$ and $r_2 > r_1 > 0$.

- (e) A **punctured disc** is a disc from which the centre point has been removed. A **punctured plane** is \mathbb{C} with a single point removed.
- (f) A **ray** or **half-line** is a set of the form

$$\{z : \text{Arg}(z - \alpha) = \theta\},$$

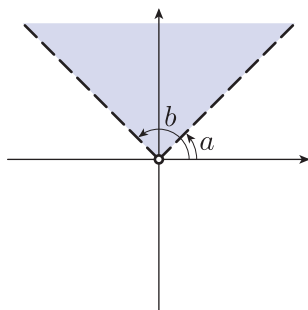
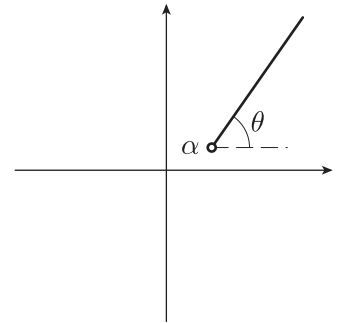
where $\alpha \in \mathbb{C}$ and $-\pi < \theta \leq \pi$.

- (g) A **sector** is a set bounded by two rays that share a common end point, possibly including one or both of the boundary rays. An **open sector** is a sector of one of the forms

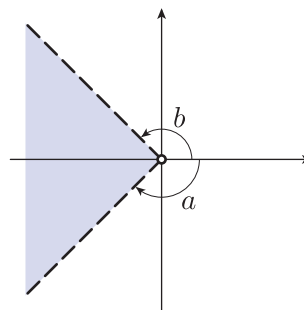
$$\{z : a < \text{Arg}(z - \alpha) < b\},$$

$$\{z : \text{Arg}(z - \alpha) < a \text{ or } \text{Arg}(z - \alpha) > b\},$$

where $\alpha \in \mathbb{C}$ and $-\pi < a < b \leq \pi$.



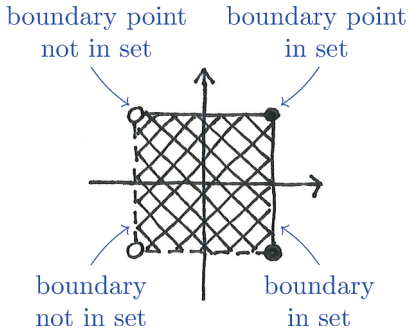
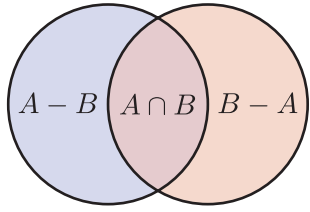
$$\{z : a < \text{Arg } z < b\}$$



$$\{z : \text{Arg } z < a \text{ or } \text{Arg } z > b\}$$

- (h) A **cut plane** is the complex plane \mathbb{C} with a half-line from the origin and the origin itself removed.

In particular, the set $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$ is a cut plane, and this set can also be specified as $\{z : |\text{Arg } z| < \pi\}$.



3. Let A and B be subsets of the complex plane.

The **union** of A and B is $A \cup B = \{z : z \in A \text{ or } z \in B\}$.

The **intersection** of A and B is $A \cap B = \{z : z \in A \text{ and } z \in B\}$.

The **difference** of A and B is $A - B = \{z : z \in A \text{ and } z \notin B\}$.

The **complement** of A is $\mathbb{C} - A = \{z : z \in \mathbb{C} \text{ and } z \notin A\}$.

4. In set notation the word ‘and’ can be replaced by a comma. For example, we can write $A \cap B = \{z : z \in A, z \in B\}$.

5. **Sketching conventions**

- The interior of a set is shown by shading (or hatching).
- Boundary curves that belong to the set are drawn unbroken.
- Boundary curves that do not belong to the set are drawn broken.
- Distinguished boundary points that belong to the set are drawn as solid dots (small, filled-in circles).
- Distinguished boundary points that do not belong to the set are drawn as hollow dots (small, empty circles).

Section 5 Proving inequalities

1. $|\operatorname{Re} z| \leq |z|$ and $|\operatorname{Im} z| \leq |z|$.
2. **Triangle Inequality** If $z_1, z_2 \in \mathbb{C}$, then
 - (a) $|z_1 + z_2| \leq |z_1| + |z_2|$ (usual form)
 - (b) $|z_1 - z_2| \geq ||z_1| - |z_2||$ (backwards form).

So $|z_1 - z_2| \geq |z_1| - |z_2|$ and $|z_1 - z_2| \geq |z_2| - |z_1|$.
3. If $z, z_1, z_2, \dots, z_n \in \mathbb{C}$, then
 - (a) $|z| \leq |\operatorname{Re} z| + |\operatorname{Im} z|$
 - (b) $|z_1 - z_2| \leq |z_1| + |z_2|$
 - (c) $|z_1 + z_2| \geq ||z_1| - |z_2||$
 - (d) $|z_1 \pm z_2 \pm \dots \pm z_n| \leq |z_1| + |z_2| + \dots + |z_n|$
 - (e) $|z_1 \pm z_2 \pm \dots \pm z_n| \geq |z_1| - |z_2| - \dots - |z_n|$.

Unit A2 Complex functions

Section 1 Complex functions and their properties

1. A **complex function** f is defined by specifying
 - two sets A and B in the complex plane \mathbb{C}
 - a rule that associates with each number z in A a unique number w in B ; we write $w = f(z)$.

The set A is called the **domain** of the function f , and the set B is called the **codomain** of f .

The number w is called the **image of z under f** , or the **value of f at z** , and we say that **f maps z to w** .

Other commonly used words for function are *mapping* and *transformation*.

2. **Convention** When a function f is specified *just* by its rule, it is to be understood that the domain of f is the set of all complex numbers to which the rule is applicable, and the codomain of f is \mathbb{C} .
3. Given a function $f: A \rightarrow B$, the **image set** of f , written $f(A)$, is the set of all values $f(z)$, where $z \in A$. Thus

$$f(A) = \{f(z) : z \in A\}.$$

If $f(A) = B$, then the function f is said to be **onto**.

4. A function $f: A \rightarrow B$ is called a **real-valued function** (of a complex variable) if $f(A) \subseteq \mathbb{R}$.

The function f is called a **real function** if $A \subseteq \mathbb{R}$ and $B \subseteq \mathbb{R}$.

5. The points at which a function takes the value zero are called the **zeros** of the function.
6. Let $f: A \rightarrow \mathbb{C}$ and $g: B \rightarrow \mathbb{C}$ be functions.

The **sum** $f + g$ is the function with domain $A \cap B$ and rule

$$(f + g)(z) = f(z) + g(z).$$

The **multiple** λf , where $\lambda \in \mathbb{C}$, is the function with domain A and rule

$$(\lambda f)(z) = \lambda f(z).$$

The **product** fg is the function with domain $A \cap B$ and rule

$$(fg)(z) = f(z)g(z).$$

The **quotient** f/g is the function with domain $A \cap B - \{z : g(z) = 0\}$ and rule

$$(f/g)(z) = f(z)/g(z).$$

7. A **polynomial function** of **degree** n is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, a_1, \dots, a_n \in \mathbb{C}$ and $a_n \neq 0$.

A **rational function** is a function of the form

$$f(z) = \frac{p(z)}{q(z)},$$

where p and q are polynomial functions, and q is not the zero function.

8. Let $f: A \rightarrow \mathbb{C}$ and $g: B \rightarrow \mathbb{C}$ be complex functions. Then the **composite function** $g \circ f$ has domain

$$\{z \in A : f(z) \in B\}$$

and rule

$$(g \circ f)(z) = g(f(z)).$$

9. The function $f: A \rightarrow B$ is **one-to-one** if the images under f of distinct points in A are also distinct; that is,

$$\text{if } z_1, z_2 \in A \text{ and } z_1 \neq z_2, \text{ then } f(z_1) \neq f(z_2).$$

An equivalent statement is that if $w \in f(A)$, then there is a *unique* z in A such that $f(z) = w$.

10. Let $f: A \rightarrow B$ be a one-to-one function. Then the **inverse function** f^{-1} of f is the function with domain $f(A)$ and rule

$$f^{-1}(w) = z,$$

where $w = f(z)$.

11. **Strategy for proving that an inverse function exists** To prove that a function f has an inverse function:
- *either* prove that f is one-to-one directly by showing that if $z_1 \neq z_2$, then $f(z_1) \neq f(z_2)$ (or, equivalently, $f(z_1) = f(z_2) \implies z_1 = z_2$)
 - *or* determine the image set $f(A)$ and show that for each $w \in f(A)$ there is a unique $z \in A$ such that $f(z) = w$.
12. Reducing the domain of a function (without changing the rule) gives a new function, called a **restriction** of the original function.

Section 2 Special types of complex function

1. Given a function f , the functions $\operatorname{Re} f: z \mapsto \operatorname{Re}(f(z))$ and $\operatorname{Im} f: z \mapsto \operatorname{Im}(f(z))$ are called the **real** and **imaginary parts** of f . They are real-valued functions with the same domain as f .
2. Given a function $f: A \rightarrow B$ and a subset S of A , the **image under f of S** , written $f(S)$, is

$$f(S) = \{f(z) : z \in S\}.$$

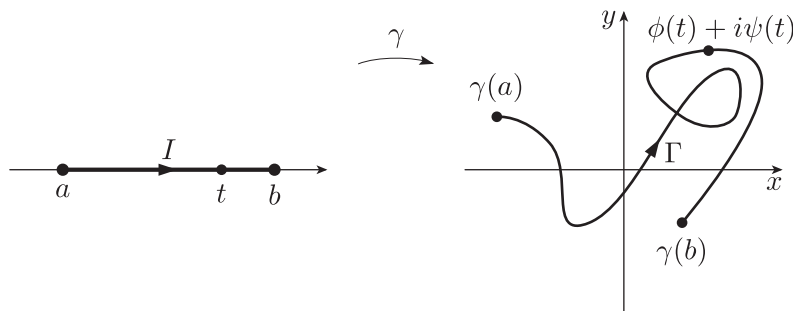
3. A **path** is a subset Γ of \mathbb{C} that is the image set of an associated continuous function $\gamma: I \rightarrow \mathbb{C}$, where I is a real interval. In this context, the function γ is called a **parametrisation** (of Γ). If

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I),$$

where ϕ and ψ are real functions, then the equations

$$x = \phi(t), \quad y = \psi(t) \quad (t \in I)$$

are called **parametric equations** (of Γ).



If I is the closed interval $[a, b]$, then $\gamma(a)$ and $\gamma(b)$ are called the **initial point** and **final point** of Γ , respectively.

The points $\gamma(a)$ and $\gamma(b)$ are also called the **endpoints** of Γ .

4. A path Γ is usually marked with an arrow (or arrows, if necessary) to show the direction in which it is traversed (the arrow points in the direction of increasing values of t).

It may be possible to obtain an equation for Γ in terms of x and y alone by eliminating t from the equations $x = \phi(t)$ and $y = \psi(t)$.

5. Let f be a continuous function, and let Γ be a path in the domain of f . Then $f(\Gamma)$ is called the **image path** (under f of Γ). If Γ has parametrisation γ , then $f(\Gamma)$ has parametrisation $f \circ \gamma$, which is the function with rule $t \mapsto f(\gamma(t))$.
6. **Strategy for determining an image path** Let f be a continuous function, and let Γ be a path with parametrisation

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I).$$

To find the image path $f(\Gamma)$:

- *either* use the geometric properties of f
- *or* substitute $x = \phi(t)$, $y = \psi(t)$ into the equation

$$u + iv = f(x + iy),$$

and then, by equating real parts and imaginary parts, obtain expressions for u and v in terms of t . (These expressions are the parametric equations of the image path $f(\Gamma)$ associated with the parametrisation $f \circ \gamma$.)

7. Standard parametrisations

Set	Standard parametrisation	Diagram
Line through α and β	$\gamma(t) = (1 - t)\alpha + t\beta \quad (t \in \mathbb{R})$	
Line segment from α to β	$\gamma(t) = (1 - t)\alpha + t\beta \quad (t \in [0, 1])$	
Circle with centre α , radius r : $ z - \alpha = r$	$\gamma(t) = \alpha + r(\cos t + i \sin t) \quad (t \in [0, 2\pi])$	
Arc of circle with centre α , radius r	$\gamma(t) = \alpha + r(\cos t + i \sin t) \quad (t \in [t_1, t_2])$	
Ellipse in standard form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a, b > 0$	$\gamma(t) = a \cos t + ib \sin t \quad (t \in [0, 2\pi])$	

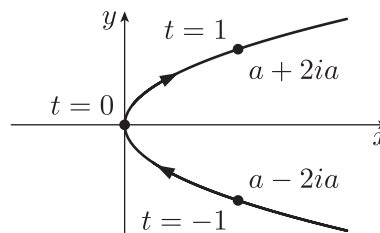
Parabola in standard

form:

$$y^2 = 4ax,$$

where $a > 0$

$$\gamma(t) = at^2 + 2iat \quad (t \in \mathbb{R})$$



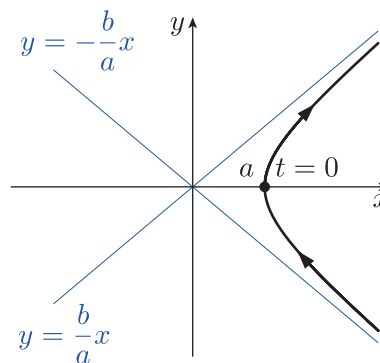
Right half of hyperbola

in standard form:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

where $a, b > 0$

$$\gamma(t) = a \cosh t + ib \sinh t \quad (t \in \mathbb{R})$$



Section 3 Images of grids

A **Cartesian grid** consists of lines of the form $x = a$ and $y = b$, usually evenly spaced in both directions.

A **polar grid** consists of circles with centre 0 and rays emerging from 0. Each of the circles has an equation of the form $r = a$, where a is a positive constant, and each of the rays has an equation of the form $\theta = b$, where b is a constant in the interval $(-\pi, \pi]$.

Section 4 Exponential, trigonometric and hyperbolic functions

1. For all $z = x + iy$ in \mathbb{C} ,

$$e^z = e^x (\cos y + i \sin y).$$

The function

$$z \mapsto e^z \quad (z \in \mathbb{C})$$

is called the **exponential function**, and is denoted by \exp .

Thus $\exp z = e^z$.

2. If z is real, $z = x + 0i$, then

$$e^z = e^x (\cos 0 + i \sin 0) = e^x.$$

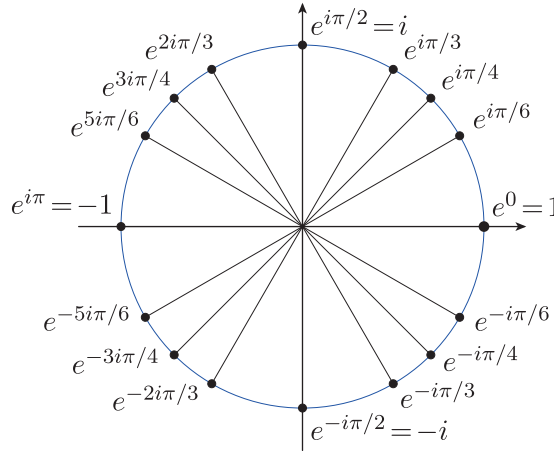
3. **Euler's Identity** If z is imaginary, $z = 0 + iy$, then

$$e^{iy} = \cos y + i \sin y.$$

4. Euler's Equation

$$e^{i\pi} + 1 = 0.$$

5. Useful values of exp on the unit circle



$$\begin{aligned} e^{i\pi/6} &= \frac{\sqrt{3}}{2} + \frac{1}{2}i \\ e^{i\pi/4} &= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\ e^{i\pi/3} &= \frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{2i\pi/3} &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{3i\pi/4} &= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i \\ e^{5i\pi/6} &= -\frac{\sqrt{3}}{2} + \frac{1}{2}i \end{aligned}$$

The formula $e^{-i\theta} = \overline{e^{i\theta}}$ can be used to find the Cartesian form of other complex numbers in the figure.

6. Exponential identities

- (a) **Addition** $e^{z_1+z_2} = e^{z_1}e^{z_2}$
- (b) **Modulus** $|e^z| = e^{\operatorname{Re} z}$
- (c) **Negatives** $e^{-z} = 1/e^z$
- (d) **Periodicity** $e^{z+2\pi i} = e^z$

7. $e^z \neq 0$ and $|e^z| \leq e^{|z|}$, for all $z \in \mathbb{C}$.

8. Given a non-zero complex number z with modulus r and argument θ , both

$$z = r(\cos \theta + i \sin \theta) \quad \text{and} \quad z = re^{i\theta}$$

are acceptable ways of writing z in polar form.

9. De Moivre's Theorem can be written as

$$(e^{i\theta})^n = e^{in\theta},$$

where n is an integer and θ is a real number.

10. The geometric nature of exp

(a) For all $n \in \mathbb{Z}$, $e^{z+2n\pi i} = e^z$. Therefore each of the points

$$z + 2n\pi i, \quad n \in \mathbb{Z},$$

has the same image under the exponential function. In particular,

$$e^z = 1 \iff z = 2n\pi i, \quad \text{for } n \in \mathbb{Z},$$

$$e^z = -1 \iff z = (2n+1)\pi i, \quad \text{for } n \in \mathbb{Z}.$$

(b) The function exp maps the line $x = a$ to the path with parametric equations

$$u = e^a \cos t, \quad v = e^a \sin t \quad (t \in \mathbb{R}).$$

This is the circle with centre 0 and radius e^a .

- (c) The function \exp maps the line $y = b$ to the path with parametric equations

$$u = e^t \cos b, \quad v = e^t \sin b \quad (t \in \mathbb{R}).$$

This is the ray from 0 (excluded) through $\cos b + i \sin b$.

- (d) The image of the strip $\{x + iy : -\pi < y \leq \pi\}$ under $f(z) = e^z$ is $\mathbb{C} - \{0\}$.

11. Trigonometric functions

For all z in \mathbb{C} ,

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz}) \quad \text{and} \quad \cos z = \frac{1}{2}(e^{iz} + e^{-iz}).$$

For all z in $\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$,

$$\tan z = \frac{\sin z}{\cos z} \quad \text{and} \quad \sec z = \frac{1}{\cos z}.$$

For all z in $\mathbb{C} - \{n\pi : n \in \mathbb{Z}\}$,

$$\cot z = \frac{\cos z}{\sin z} \quad \text{and} \quad \operatorname{cosec} z = \frac{1}{\sin z}.$$

12. Theorem

- (a) The zeros of the sine function are given by

$$\sin z = 0 \iff z = n\pi, \quad \text{for } n \in \mathbb{Z}.$$

- (b) The zeros of the cosine function are given by

$$\cos z = 0 \iff z = (n + \frac{1}{2})\pi, \quad \text{for } n \in \mathbb{Z}.$$

13. The well-known properties of the real sine and cosine functions

$$|\sin x| \leq 1 \quad \text{and} \quad |\cos x| \leq 1, \quad \text{where } x \in \mathbb{R},$$

do not always hold when x is replaced by a complex number z .

14. Trigonometric identities

All the standard identities satisfied by the real trigonometric functions (see item 4 of the introductory section on real functions) also hold for the complex trigonometric functions.

15. Hyperbolic functions

For all z in \mathbb{C} ,

$$\sinh z = \frac{1}{2}(e^z - e^{-z}) \quad \text{and} \quad \cosh z = \frac{1}{2}(e^z + e^{-z}).$$

For all z in $\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$,

$$\tanh z = \frac{\sinh z}{\cosh z} \quad \text{and} \quad \operatorname{sech} z = \frac{1}{\cosh z}.$$

For all z in $\mathbb{C} - \{n\pi i : n \in \mathbb{Z}\}$,

$$\coth z = \frac{\cosh z}{\sinh z} \quad \text{and} \quad \operatorname{cosech} z = \frac{1}{\sinh z}.$$

16. (a) The zeros of \sinh are given by

$$\sinh z = 0 \iff z = n\pi i, \quad \text{for } n \in \mathbb{Z}.$$
 (b) The zeros of \cosh are given by

$$\cosh z = 0 \iff z = \left(n + \frac{1}{2}\right)\pi i, \quad \text{for } n \in \mathbb{Z}.$$
17. **Theorem** For all z in \mathbb{C} ,

$$\sin(iz) = i \sinh z \quad \text{and} \quad \cos(iz) = \cosh z,$$

$$\sinh(iz) = i \sin z \quad \text{and} \quad \cosh(iz) = \cos z.$$
18. **Hyperbolic identities** All the standard identities satisfied by the real hyperbolic functions also hold for the complex hyperbolic functions.

Section 5 Logarithms and powers

1. For $z \in \mathbb{C} - \{0\}$, the **principal logarithm** of z is

$$\text{Log } z = \log |z| + i \text{Arg } z.$$
 The function $z \mapsto \text{Log } z$ is called the **principal logarithm function**.
2. If z is real and positive (that is, $z = x + 0i$, where $x > 0$), then

$$\text{Log } z = \text{Log } x = \log x.$$
3. Log is the inverse function of

$$f(z) = e^z \quad (z \in \{x + iy : -\pi < y \leq \pi\}),$$
 and it satisfies

$$e^{\text{Log } z} = z, \quad \text{for } z \in \mathbb{C} - \{0\},$$

$$\text{Log}(e^z) = z, \quad \text{for } z \in \{x + iy : -\pi < y \leq \pi\}.$$
4. **Logarithmic identities**
- (a) **Multiplication**

$$\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2, \quad \text{if } \text{Arg } z_1, \text{Arg } z_2 \in (-\pi/2, \pi/2].$$
- (b) **Reciprocals**

$$\text{Log}(1/z) = -\text{Log } z, \quad \text{if } \text{Arg } z \in (-\pi, \pi).$$
5. Item 4(a) holds in the following form for any $z_1, z_2 \in \mathbb{C} - \{0\}$:

$$\text{Log}(z_1 z_2) = \text{Log } z_1 + \text{Log } z_2 + 2n\pi i,$$
 where n is -1 , 0 or 1 depending on whether $\text{Arg } z_1 + \text{Arg } z_2$ is greater than π , lies in the interval $(-\pi, \pi]$, or is less than or equal to $-\pi$.
6. **The geometric nature of Log**
- (a) The circle centred at 0 of radius r is mapped to the line segment
 $u = \log r, \quad -\pi < v \leq \pi$ (where $w = u + iv$).
- (b) The ray $\text{Arg } z = \theta$ is mapped to the horizontal line $v = \theta$.
7. For $z, \alpha \in \mathbb{C}$, with $z \neq 0$, the **principal α th power of z** is

$$z^\alpha = \exp(\alpha \text{Log } z).$$
 (This agrees with the usual meaning of z^α for $\alpha = n$ or $\alpha = 1/n$, where $n \in \mathbb{N}$.)
 The function $z \mapsto z^\alpha$ is called the **principal α th power function**.

Unit A3 Continuity

Section 1 Sequences

1. A **(complex) sequence** is an unending list of complex numbers

$$z_1, z_2, z_3, \dots$$

The complex number z_n is called the **n th term of the sequence** and the sequence is denoted by (z_n) .

2. The sequence (z_n) is **convergent with limit α** , or **converges to α** , or **tends to α** , if for each positive number ε , there is an integer N such that

$$|z_n - \alpha| < \varepsilon, \quad \text{for all } n > N.$$

If (z_n) converges to α , then we write

- either $\lim_{n \rightarrow \infty} z_n = \alpha$
- or $z_n \rightarrow \alpha$ as $n \rightarrow \infty$.

If the limit α is 0, then (z_n) is called a **null sequence**.

3. (a) If a sequence (z_n) is convergent, then it has a *unique* limit.
 (b) If a sequence converges to α , then this remains true if we add, delete or alter a finite number of terms.
4. The sequence (z_n) converges to α if and only if $(z_n - \alpha)$ is a null sequence. That is,

$$z_n \rightarrow \alpha \text{ as } n \rightarrow \infty \iff z_n - \alpha \rightarrow 0 \text{ as } n \rightarrow \infty.$$

5. A sequence (z_n) is **constant** if there is a number α with $z_n = \alpha$, $n = 1, 2, \dots$, in which case the sequence converges to α .
6. **Squeeze Rule** If (a_n) is a real null sequence of non-negative terms, and if

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \dots,$$

then (z_n) is a null sequence.

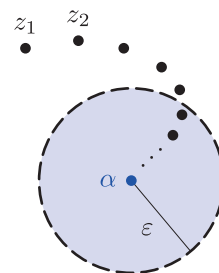
7. When the inequality

$$|z_n| \leq a_n$$

holds for $n = 1, 2, \dots$ (or even for all but a finite number of terms of the sequence), we say that the real sequence (a_n) **dominates** the sequence (z_n) .

8. **Basic null sequences** The following sequences are null:

- (a) $\left(\frac{1}{n^p}\right)$, for $p > 0$
 (b) (α^n) , for $|\alpha| < 1$.



9. **Combination Rules for Sequences** If $\lim_{n \rightarrow \infty} z_n = \alpha$ and $\lim_{n \rightarrow \infty} w_n = \beta$, then

(a) **Sum Rule** $\lim_{n \rightarrow \infty} (z_n + w_n) = \alpha + \beta$

(b) **Multiple Rule** $\lim_{n \rightarrow \infty} (\lambda z_n) = \lambda \alpha$, where $\lambda \in \mathbb{C}$

(c) **Product Rule** $\lim_{n \rightarrow \infty} (z_n w_n) = \alpha \beta$

(d) **Quotient Rule** $\lim_{n \rightarrow \infty} \left(\frac{z_n}{w_n} \right) = \frac{\alpha}{\beta}$, provided that $\beta \neq 0$.

10. **Theorem** If $\lim_{n \rightarrow \infty} z_n = \alpha$, then

(a) $\lim_{n \rightarrow \infty} |z_n| = |\alpha|$

(b) $\lim_{n \rightarrow \infty} \overline{z_n} = \overline{\alpha}$

(c) $\lim_{n \rightarrow \infty} \operatorname{Re} z_n = \operatorname{Re} \alpha$

(d) $\lim_{n \rightarrow \infty} \operatorname{Im} z_n = \operatorname{Im} \alpha$.

11. A sequence that is not convergent is **divergent**.

12. The sequence (z_n) **tends to infinity** if, for each positive number M , there is an integer N such that

$$|z_n| > M, \quad \text{for all } n > N.$$

In this case we write

$$z_n \rightarrow \infty \text{ as } n \rightarrow \infty.$$

13. **Reciprocal Rule for Sequences** Let (z_n) be a sequence. Then

$$z_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

if and only if

$$(1/z_n) \text{ is a null sequence.}$$

14. Let (n_k) be a sequence of positive integers that is strictly increasing; that is, $n_1 < n_2 < n_3 < \dots$.

Then the sequence (z_{n_k}) is a **subsequence** of the sequence (z_n) .

15. In particular, (z_{2k}) is the **even subsequence** and (z_{2k-1}) is the **odd subsequence** of (z_n) .

16. **Subsequence Rules**

(a) **First Subsequence Rule** The sequence (z_n) is divergent if (z_n) has two convergent subsequences with different limits.

(b) **Second Subsequence Rule** The sequence (z_n) is divergent if (z_n) has a subsequence that tends to infinity.

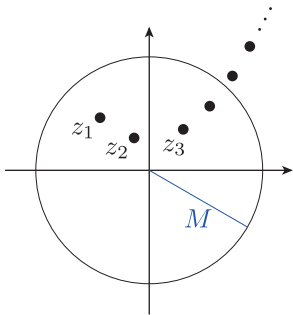
17. **Theorem**

(a) If $|\alpha| > 1$, then the sequence (α^n) tends to infinity.

(b) If $|\alpha| = 1$ and $\alpha \neq 1$, then the sequence (α^n) is divergent.

18. A sequence (z_n) is **bounded** if there is a positive number M such that $|z_n| \leq M$, for $n = 1, 2, \dots$.

Every convergent sequence is bounded, but not every bounded sequence is convergent.



Section 2 Continuous functions

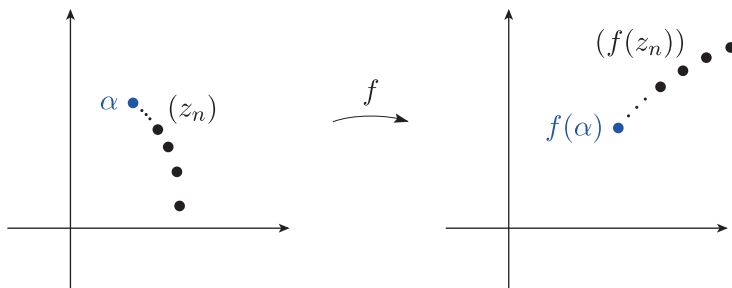
1. **Continuity: sequential definition** Let $f: A \rightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each sequence (z_n) in A such that $z_n \rightarrow \alpha$, we have

$$f(z_n) \rightarrow f(\alpha);$$

that is,

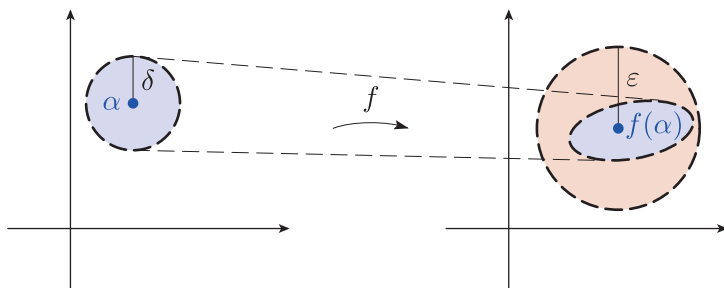
$$z_n \rightarrow \alpha \implies f(z_n) \rightarrow f(\alpha).$$

If f is continuous at each α in A , then we say that f is **continuous** (on A).



2. Let $f: A \rightarrow \mathbb{C}$ and $\alpha \in A$. If f is not continuous at α , then we say that f is **discontinuous at α** .
3. **Continuity: ε - δ definition** Let $f: A \rightarrow \mathbb{C}$ and $\alpha \in A$. Then f is **continuous at α** if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - f(\alpha)| < \varepsilon, \quad \text{for all } z \in A \text{ with } |z - \alpha| < \delta.$$



4. **Theorem** The ε - δ definition of continuity is equivalent to the sequential definition of continuity.
5. **Strategy for determining whether a function is continuous**
To determine whether a function $f: A \rightarrow \mathbb{C}$ is continuous at a point α in A , apply the following steps.

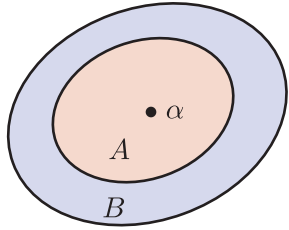
- (1) Guess whether f is continuous or discontinuous at α .
- (2) If you believe that f is continuous at α , then check that

$$z_n \rightarrow \alpha \implies f(z_n) \rightarrow f(\alpha)$$

for *every* sequence (z_n) in A that tends to α .

- (3) If you believe that f is discontinuous at α , then find *just one* sequence (z_n) in A such that $z_n \rightarrow \alpha$ but $f(z_n) \not\rightarrow f(\alpha)$.

In step 2 you may choose to use the ε - δ definition of continuity instead of the sequential definition.



6. **Combination Rules for Continuous Functions** Let f and g be functions that are continuous at α .
 - (a) **Sum Rule** $f + g$ is continuous at α .
 - (b) **Multiple Rule** λf is continuous at α , for $\lambda \in \mathbb{C}$.
 - (c) **Product Rule** fg is continuous at α .
 - (d) **Quotient Rule** f/g is continuous at α , provided that $g(\alpha) \neq 0$.
7. **Composition Rule for Continuous Functions** Let f be a function that is continuous at α , and let g be a function that is continuous at $f(\alpha)$. Then $g \circ f$ is continuous at α .
8. **Restriction Rule for Continuous Functions** Let f and g be complex functions with domains A and B , respectively, and let $A \subseteq B$. If
 - $f(z) = g(z)$, for $z \in A$
 - g is continuous at $\alpha \in A$,
 then f is continuous at α .
9. **Basic continuous functions** The following functions are continuous:
 - (a) polynomial and rational functions
 - (b) $f(z) = |z|, \bar{z}, \operatorname{Re} z, \operatorname{Im} z$
 - (c) $f(z) = e^z$
 - (d) trigonometric and hyperbolic functions
 - (e) $f(z) = \operatorname{Arg} z, \operatorname{Log} z, z^\alpha$, on $\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$.
10. The functions $f(z) = \operatorname{Arg} z, \operatorname{Log} z, z^\alpha$, for $\alpha \in \mathbb{C} - \mathbb{Z}$, are discontinuous at each point of $\{x \in \mathbb{R} : x < 0\}$.

Section 3 Limits of functions

1. The point α is a **limit point** of a set A in \mathbb{C} if there is a sequence (z_n) in $A - \{\alpha\}$ such that

$$\lim_{n \rightarrow \infty} z_n = \alpha.$$

2. **Limit: sequential definition** Let f be a function with domain A , and suppose that α is a limit point of A . Then the function f has **limit β as z tends to α** if, for each sequence (z_n) in $A - \{\alpha\}$ such that $z_n \rightarrow \alpha$, we have

$$f(z_n) \rightarrow \beta.$$

In this case we write

- either $\lim_{z \rightarrow \alpha} f(z) = \beta$
- or $f(z) \rightarrow \beta$ as $z \rightarrow \alpha$,

and we say that the **limit exists**.

3. Since the sequences considered lie in $A - \{\alpha\}$, the value $f(\alpha)$ need not be defined in order for $\lim_{z \rightarrow \alpha} f(z)$ to exist. Even when $f(\alpha)$ is defined, its value has no bearing on the existence or the value of this limit.
4. **Limit: ε - δ definition** Let $f: A \rightarrow \mathbb{C}$ and suppose that α is a limit point of A . Then the function f has **limit β as z tends to α** if, for each $\varepsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - \beta| < \varepsilon, \quad \text{for all } z \in A - \{\alpha\} \text{ with } |z - \alpha| < \delta.$$

5. The ε - δ definition of the limit of a function is equivalent to the sequential definition of the limit of a function.
6. **Strategy for proving that a limit does not exist** To prove that $\lim_{z \rightarrow \alpha} f(z)$ does not exist, where α is a limit point of the domain A of the function f :
- *either* find two sequences (z_n) and (z'_n) in $A - \{\alpha\}$ that both tend to α such that the sequences $(f(z_n))$ and $(f(z'_n))$ have different limits
 - *or* find a sequence (z_n) in $A - \{\alpha\}$ that tends to α such that the sequence $(f(z_n))$ tends to infinity.

7. **Theorem** Let f be a function with domain A and suppose that the point $\alpha \in A$ is a limit point of A . Then

$$f \text{ is continuous at } \alpha \iff \lim_{z \rightarrow \alpha} f(z) = f(\alpha).$$

8. **Combination Rules for Limits of Functions** Let f and g be functions with domains A and B , respectively, and suppose that α is a limit point of $A \cap B$. If

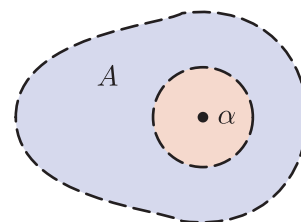
$$\lim_{z \rightarrow \alpha} f(z) = \beta \quad \text{and} \quad \lim_{z \rightarrow \alpha} g(z) = \gamma,$$

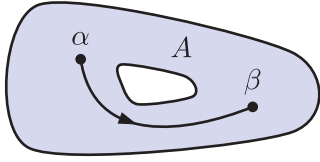
then

- (a) **Sum Rule** $\lim_{z \rightarrow \alpha} (f(z) + g(z)) = \beta + \gamma$
- (b) **Multiple Rule** $\lim_{z \rightarrow \alpha} (\lambda f(z)) = \lambda\beta, \quad \text{for } \lambda \in \mathbb{C}$
- (c) **Product Rule** $\lim_{z \rightarrow \alpha} (f(z)g(z)) = \beta\gamma$
- (d) **Quotient Rule** $\lim_{z \rightarrow \alpha} (f(z)/g(z)) = \beta/\gamma, \quad \text{provided that } \gamma \neq 0.$

Section 4 Regions

1. A set A in \mathbb{C} is **open** if each point α in A is the centre of some open disc lying entirely in A .
2. The empty set \emptyset is open, as is the set \mathbb{C} . Any open half-plane or open disc is an open set.
3. **Combination Rules for Open Sets** If A_1 and A_2 are open sets, then so are
- (a) $A_1 \cup A_2$
 - (b) $A_1 \cap A_2$.
4. If A_1, A_2, \dots, A_n are open sets, then so are
- (a) $A_1 \cup A_2 \cup \dots \cup A_n$
 - (b) $A_1 \cap A_2 \cap \dots \cap A_n$.





5. A set A in \mathbb{C} is **(pathwise) connected** if any two distinct points α and β in A can be joined by a path lying entirely in A .
6. A connected set in which any two distinct points α and β can be joined by a line segment that lies entirely within the set is called **convex**.
7. **Theorem** Let f be a continuous function whose domain A is connected. Then the image set $f(A)$ is also connected.
8. A **region** is a non-empty, connected, open subset of \mathbb{C} .
9. **Basic regions** The following subsets of \mathbb{C} are regions:
 - any open disc
 - any open half-plane
 - the complement of any closed disc
 - any open annulus
 - any open rectangle
 - any open sector (including cut planes)
 - the set \mathbb{C} itself.
10. **Theorem** If \mathcal{R} is a region and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathcal{R}$, then $\mathcal{R} - \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is also a region.

Section 5 The Extreme Value Theorem

1. A set E in \mathbb{C} is **closed** if its complement $\mathbb{C} - E$ is open.
2. The empty set \emptyset is closed, as is the set \mathbb{C} . Every single element set $\{\alpha\}$ is closed. Any closed half-plane or closed disc is a closed set.
3. If E is a closed set and (z_n) is a convergent sequence in E with limit α , then $\alpha \in E$.
4. **Combination Rules for Closed Sets** If E_1 and E_2 are closed sets, then so are
 - (a) $E_1 \cup E_2$
 - (b) $E_1 \cap E_2$.
5. If E_1, E_2, \dots, E_n are closed sets, then so are
 - (a) $E_1 \cup E_2 \cup \dots \cup E_n$
 - (b) $E_1 \cap E_2 \cap \dots \cap E_n$.
6. **Warning!** If a set contains some but not all of its boundary points, then it is *neither* open *nor* closed. The sets \mathbb{C} and \emptyset are the only subsets of \mathbb{C} that are both open and closed.
7. A set E in \mathbb{C} is **bounded** if it is contained in some closed disc. A set is **unbounded** if it is not bounded.
8. A set E in \mathbb{C} is **compact** if it is closed and bounded.
9. **Extreme Value Theorem** Let f be a function that is continuous on a compact set E . Then there are numbers α and β in E such that

$$|f(\beta)| \leq |f(z)| \leq |f(\alpha)|, \quad \text{for all } z \in E.$$

10. A function f whose domain contains a set E is said to be **bounded** on E if the set $f(E)$ is a bounded set. If f is not bounded on E , then it is said to be **unbounded** on E .

11. **Boundedness Theorem** Let f be a function that is continuous on a compact set E . Then there is a number M such that

$$|f(z)| \leq M, \quad \text{for all } z \in E.$$

12. **Theorem** Let f be a function that is continuous on a compact set E . Then $f(E)$ is compact.

13. Let A be a subset of \mathbb{C} , and let $\alpha \in \mathbb{C}$. Then

- α is an **interior point** of A if there is an open disc centred at α that lies entirely in A
- α is an **exterior point** of A if there is an open disc centred at α that lies entirely outside A .

The set of interior points of A forms the **interior** $\text{int } A$ of A , and the set of exterior points of A forms the **exterior** $\text{ext } A$ of A .

14. Let A be a subset of \mathbb{C} and let $\alpha \in \mathbb{C}$. Then α is a **boundary point** of A if each open disc centred at α contains at least one point of A and at least one point of $\mathbb{C} - A$.

The set of boundary points of A forms the **boundary** ∂A of A .

15. The sets $\text{int } A$, $\text{ext } A$ and ∂A are disjoint and

$$\partial A = \mathbb{C} - (\text{int } A \cup \text{ext } A).$$

16. **Theorem** If A is a subset of \mathbb{C} , then

- (a) $\text{int } A$ and $\text{ext } A$ are open
- (b) ∂A is closed.

Unit A4 Differentiation

Section 1 Derivatives of complex functions

1. Let f be a complex function whose domain contains the point α . Then the **derivative of f at α** is

$$\lim_{z \rightarrow \alpha} \frac{f(z) - f(\alpha)}{z - \alpha} \quad \left(\text{or } \lim_{h \rightarrow 0} \frac{f(\alpha + h) - f(\alpha)}{h} \right),$$

provided that this limit exists. If it does exist, then f is **differentiable at α** . If f is differentiable at *every* point of a set A , then f is **differentiable on A** . A function is **differentiable** if it is differentiable on its domain.

The derivative of f at α is denoted by $f'(\alpha)$, and the function

$$f': z \mapsto f'(z)$$

is called the **derivative of the function f** . The domain of f' is the set of all complex numbers at which f is differentiable.

2. The derivative $f'(z)$ is sometimes written as $\frac{df}{dz}(z)$ or $\frac{d}{dz}(f(z))$.
3. Let f be a complex function. The **higher-order derivatives of f** are obtained by repeated differentiation:

$$(f')' = f'' = f^{(2)}, \quad (f'')' = f''' = f^{(3)}, \quad \text{and so on.}$$

The **n th derivative of f** is the function $f^{(n)}$.

4. A function is **entire** if it is differentiable on the whole of \mathbb{C} .
5. Polynomial functions, \exp , \sin , \cos , \sinh and \cosh are all entire functions. \log , \tan , \tanh and $z \mapsto z^\alpha$, for $\alpha \neq 0, 1, 2, \dots$, are not entire functions; each of them is differentiable only on a proper subset of \mathbb{C} .
6. A function that is differentiable on a region \mathcal{R} is said to be **analytic on \mathcal{R}** . If the domain of a function f is a region, and if f is differentiable on its domain, then f is said to be **analytic**. A function is **analytic at a point α** if it is differentiable on a region containing α .
7. **Theorem** Let f be a complex function that is differentiable at α . Then f is continuous at α .
8. **Linear Approximation Theorem** Let f be a complex function that is differentiable at α . Then f can be approximated near α by a linear polynomial. More precisely,

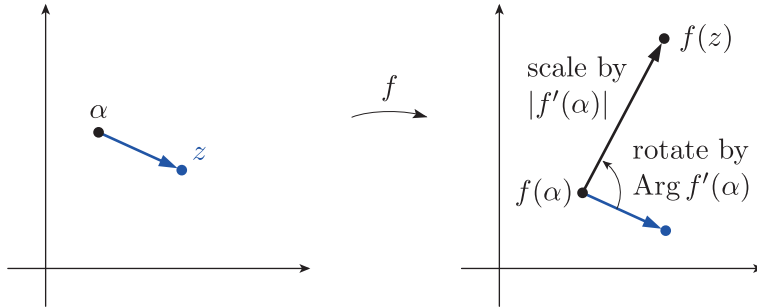
$$f(z) = f(\alpha) + (z - \alpha)f'(\alpha) + e(z),$$

where e is an ‘error function’ satisfying $e(z)/(z - \alpha) \rightarrow 0$ as $z \rightarrow \alpha$.

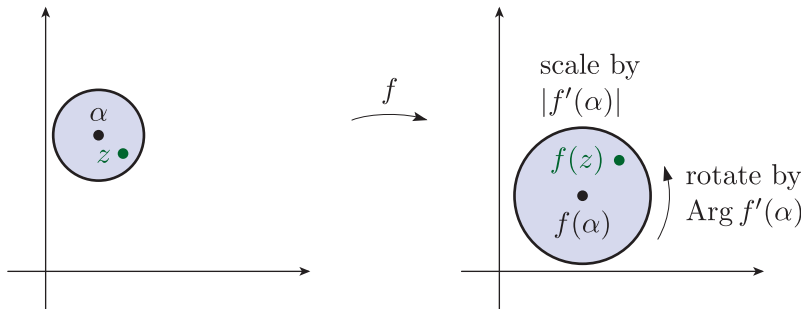
9. **Geometric interpretation of derivatives** If $f'(\alpha) \neq 0$, then, to a close approximation,

$$f(z) - f(\alpha) \approx f'(\alpha)(z - \alpha), \quad \text{for } z \text{ near to } \alpha.$$

Multiplication of $z - \alpha$ by $f'(\alpha)$ has the effect of scaling $z - \alpha$ by the factor $|f'(\alpha)|$ and rotating it about 0 through the angle $\text{Arg } f'(\alpha)$.



To a close approximation, f maps a small disc centred at α to a small disc centred at $f(\alpha)$. In the process, the disc is scaled by the factor $|f'(\alpha)|$, and rotated about its centre through the angle $\text{Arg } f'(\alpha)$.



10. **Combination Rules for Differentiation** Let f and g be complex functions with domains A and B , respectively, and let α be a limit point of $A \cap B$. If f and g are differentiable at α , then

- (a) **Sum Rule** $f + g$ is differentiable at α , and

$$(f + g)'(\alpha) = f'(\alpha) + g'(\alpha)$$

- (b) **Multiple Rule** λf is differentiable at α , for $\lambda \in \mathbb{C}$, and

$$(\lambda f)'(\alpha) = \lambda f'(\alpha)$$

- (c) **Product Rule** fg is differentiable at α , and

$$(fg)'(\alpha) = f'(\alpha)g(\alpha) + f(\alpha)g'(\alpha)$$

- (d) **Quotient Rule** f/g is differentiable at α (provided that $g(\alpha) \neq 0$), and

$$\left(\frac{f}{g}\right)'(\alpha) = \frac{g(\alpha)f'(\alpha) - f(\alpha)g'(\alpha)}{(g(\alpha))^2}.$$

11. **Reciprocal Rule for Differentiation** Let f be a function that is differentiable at α . If $f(\alpha) \neq 0$, then $1/f$ is differentiable at α , and

$$\left(\frac{1}{f}\right)'(\alpha) = -\frac{f'(\alpha)}{(f(\alpha))^2}.$$

12. **Differentiating Polynomial Functions** Let p be the polynomial function

$$p(z) = a_n z^n + \cdots + a_2 z^2 + a_1 z + a_0 \quad (z \in \mathbb{C}),$$

where $a_0, a_1, \dots, a_n \in \mathbb{C}$ and $a_n \neq 0$. Then p is entire with derivative

$$p'(z) = n a_n z^{n-1} + \cdots + 2 a_2 z + a_1 \quad (z \in \mathbb{C}).$$

13. Rational functions are analytic.

14. **Strategy A for non-differentiability** If f is discontinuous at α , then f is not differentiable at α .

15. **Strategy B for non-differentiability** To prove that a function f is not differentiable at α , apply the strategy for proving that a limit does not exist (item 6 in Section 3 of Unit A3) to the difference quotient

$$\frac{f(z) - f(\alpha)}{z - \alpha}.$$

Section 2 The Cauchy–Riemann equations

1. Let $u: A \rightarrow \mathbb{R}$ be a function whose domain A is a subset of \mathbb{R}^2 that contains the point (a, b) .

- The **partial derivative of u with respect to x at (a, b)** , denoted $\frac{\partial u}{\partial x}(a, b)$, is the derivative of the function $x \mapsto u(x, b)$ at $x = a$, provided that this derivative exists.
- The **partial derivative of u with respect to y at (a, b)** , denoted $\frac{\partial u}{\partial y}(a, b)$, is the derivative of the function $y \mapsto u(a, y)$ at $y = b$, provided that this derivative exists.

2. **Cauchy–Riemann Theorem** Let $f(x + iy) = u(x, y) + iv(x, y)$ be defined on a region \mathcal{R} containing $a + ib$.

If f is differentiable at $a + ib$, then

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist at (a, b) and satisfy the **Cauchy–Riemann equations**

$$\frac{\partial u}{\partial x}(a, b) = \frac{\partial v}{\partial y}(a, b) \quad \text{and} \quad \frac{\partial v}{\partial x}(a, b) = -\frac{\partial u}{\partial y}(a, b).$$

3. **Strategy C for non-differentiability**

Let $f(x + iy) = u(x, y) + iv(x, y)$. If either

$$\frac{\partial u}{\partial x}(a, b) \neq \frac{\partial v}{\partial y}(a, b) \quad \text{or} \quad \frac{\partial v}{\partial x}(a, b) \neq -\frac{\partial u}{\partial y}(a, b),$$

then f is not differentiable at $a + ib$.

4. Cauchy–Riemann Converse Theorem

Let $f(x + iy) = u(x, y) + iv(x, y)$ be defined on a region \mathcal{R} containing $a + ib$. If the partial derivatives $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$

- exist at (x, y) for each $x + iy \in \mathcal{R}$
- are continuous at (a, b)
- satisfy the Cauchy–Riemann equations at (a, b) ,

then f is differentiable at $a + ib$ and

$$f'(a + ib) = \frac{\partial u}{\partial x}(a, b) + i \frac{\partial v}{\partial x}(a, b).$$

Section 3 Rules for manipulating differentiable functions

1. **Chain Rule** Let f and g be complex functions, and let α be a limit point of the domain of $g \circ f$. If f is differentiable at α , and g is differentiable at $f(\alpha)$, then $g \circ f$ is differentiable at α , and

$$(g \circ f)'(\alpha) = g'(f(\alpha))f'(\alpha).$$

2. **Inverse Function Rule** Let $f: A \rightarrow B$ be a one-to-one complex function, and suppose that f^{-1} is continuous at $\beta \in B$. If f has a non-zero derivative at $f^{-1}(\beta) \in A$, then f^{-1} is differentiable at β and

$$(f^{-1})'(\beta) = \frac{1}{f'(f^{-1}(\beta))}.$$

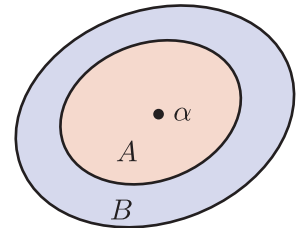
3. **Restriction Rule for Differentiation** Let f and g be complex functions with domains A and B , respectively, and let $A \subseteq B$. If $\alpha \in A$ is a limit point of A and

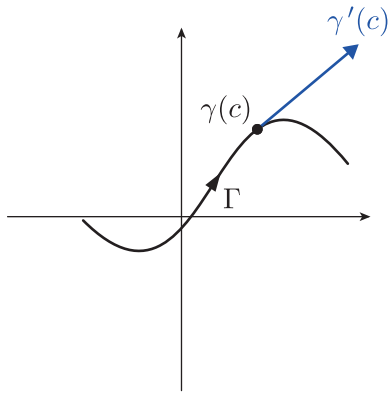
- $f(z) = g(z)$, for $z \in A$
- g is differentiable at α ,

then f is differentiable at α , and $f'(\alpha) = g'(\alpha)$.

4. **Standard derivatives**

$f(z)$	$f'(z)$	Domain of f'
$\alpha, \alpha \in \mathbb{C}$	0	\mathbb{C}
$z^k, k \in \mathbb{Z}, k > 0$	kz^{k-1}	\mathbb{C}
$z^k, k \in \mathbb{Z}, k < 0$	kz^{k-1}	$\mathbb{C} - \{0\}$
$z^\alpha, \alpha \in \mathbb{C} - \mathbb{Z}$	$\alpha z^{\alpha-1}$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\exp z$	$\exp z$	\mathbb{C}
$\text{Log } z$	$1/z$	$\mathbb{C} - \{x \in \mathbb{R} : x \leq 0\}$
$\sin z$	$\cos z$	\mathbb{C}
$\cos z$	$-\sin z$	\mathbb{C}
$\tan z$	$\sec^2 z$	$\mathbb{C} - \{(n + \frac{1}{2})\pi : n \in \mathbb{Z}\}$
$\sinh z$	$\cosh z$	\mathbb{C}
$\cosh z$	$\sinh z$	\mathbb{C}
$\tanh z$	$\text{sech}^2 z$	$\mathbb{C} - \{(n + \frac{1}{2})\pi i : n \in \mathbb{Z}\}$





Section 4 Smooth paths

1. **Tangent vectors to paths** Let Γ be a path with parametrisation $\gamma: I \rightarrow \mathbb{C}$, and suppose that $c \in I$. If γ is differentiable at c and if $\gamma'(c) \neq 0$, then $\gamma'(c)$ can be interpreted geometrically as a *tangent vector* to the path Γ at the point $\gamma(c)$.

2. **Theorem** Let ϕ and ψ be real functions, both with domain an interval I . Then the parametrisation

$$\gamma(t) = \phi(t) + i\psi(t) \quad (t \in I)$$

is differentiable at a point $c \in I$ if and only if both ϕ and ψ are differentiable at c . If ϕ and ψ are differentiable at c , then

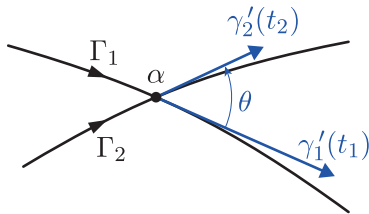
$$\gamma'(c) = \phi'(c) + i\psi'(c).$$

3. A parametrisation $\gamma: I \rightarrow \mathbb{C}$ is **smooth** if

- γ is differentiable on I
- γ' is continuous on I
- γ' is non-zero on I .

A path is **smooth** if it has a smooth parametrisation.

4. **Images of tangent vectors** Let f be a function that is analytic on a region \mathcal{R} , and suppose that $f'(\alpha) \neq 0$ for some $\alpha \in \mathcal{R}$. If Γ is a smooth path in \mathcal{R} that passes through α , then the tangent vector to the image path $f(\Gamma)$ at $f(\alpha)$ can be obtained from the tangent vector to Γ at α by a rotation through the angle $\text{Arg } f'(\alpha)$ and a scaling by the factor $|f'(\alpha)|$.



5. Suppose that Γ_1 and Γ_2 are two smooth paths with parametrisations $\gamma_1: I_1 \rightarrow \mathbb{C}$ and $\gamma_2: I_2 \rightarrow \mathbb{C}$ that intersect at the point $\alpha = \gamma_1(t_1) = \gamma_2(t_2)$. Then the **angle from Γ_1 to Γ_2 at α** is

$$\theta = \text{Arg} \left(\frac{\gamma_2'(t_2)}{\gamma_1'(t_1)} \right).$$

6. A function that is analytic at a point α is said to be **conformal at α** if the angle from any smooth path through α to any other smooth path through α is preserved by the function. A function is **conformal on a set S** if it is conformal at every point of S . A function is **conformal** if it is conformal on its domain, in which case it is called a **conformal mapping**.
7. **Theorem** Let f be a function that is analytic at a point α . Then f is conformal at α if and only if $f'(\alpha) \neq 0$.
8. Smooth paths that meet at right angles are said to be **orthogonal**. An **orthogonal grid** is a grid made up of orthogonal smooth paths.

Unit B1 Integration

Section 1 revises the definition and main properties of the integration of real functions, and motivates the integration of complex functions.

Section 2 Integrating complex functions

1. Let $f: [a, b] \rightarrow \mathbb{C}$ be a complex function with real part $u = \operatorname{Re} f$ and imaginary part $v = \operatorname{Im} f$, so $f(t) = u(t) + iv(t)$, for $t \in [a, b]$. Then

$$\int_a^b f(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

2. Let $\Gamma: \gamma(t)$ ($t \in [a, b]$) be a smooth path in \mathbb{C} , and let f be a function that is continuous on Γ . Then the **integral of f along the path Γ** , denoted by $\int_{\Gamma} f(z) dz$ or $\int_{\Gamma} f$, is

$$\int_{\Gamma} f(z) dz = \int_a^b f(\gamma(t)) \gamma'(t) dt.$$

The integral is evaluated by splitting $f(\gamma(t)) \gamma'(t)$ into its real and imaginary parts $u(t) = \operatorname{Re}(f(\gamma(t)) \gamma'(t))$ and $v(t) = \operatorname{Im}(f(\gamma(t)) \gamma'(t))$, and evaluating the resulting pair of real integrals,

$$\int_a^b f(\gamma(t)) \gamma'(t) dt = \int_a^b u(t) dt + i \int_a^b v(t) dt.$$

3. A convenient way to remember this definition is to write

$$z = \gamma(t), \quad dz = \gamma'(t) dt.$$

4. **Theorem** Let $\gamma_1: [a_1, b_1] \rightarrow \mathbb{C}$ and $\gamma_2: [a_2, b_2] \rightarrow \mathbb{C}$ be two smooth parametrisations of paths with the same initial point, final point and image set Γ such that γ_1 and γ_2 are one-to-one on $[a_1, b_1]$ and $[a_2, b_2]$, respectively. Let f be a function that is continuous on Γ . Then

$$\int_{\Gamma} f(z) dz$$

does not depend on which parametrisation γ_1 or γ_2 is used.

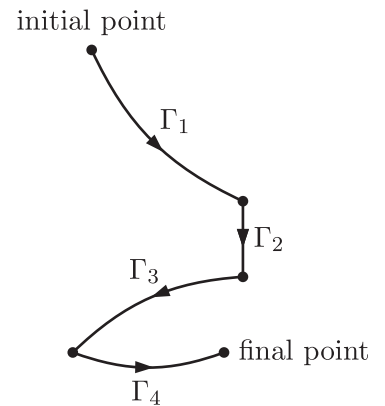
5. A **contour** Γ is a path that can be subdivided into a finite number of smooth paths $\Gamma_1, \Gamma_2, \dots, \Gamma_n$ joined end to end. The order of these constituent smooth paths is indicated by writing

$$\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n.$$

The **initial point** of Γ is the initial point of Γ_1 , and the **final point** of Γ is the final point of Γ_n .

6. Let $\Gamma = \Gamma_1 + \Gamma_2 + \dots + \Gamma_n$ be a contour, and let f be a function that is continuous on Γ . Then the **(contour) integral of f along Γ** , denoted by $\int_{\Gamma} f(z) dz$ or $\int_{\Gamma} f$, is

$$\int_{\Gamma} f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \dots + \int_{\Gamma_n} f(z) dz.$$



7. The value of a contour integral is independent of the way that the contour is split into smooth paths.
8. **Combination Rules for Contour Integrals** Let Γ be a contour, and let f and g be functions that are continuous on Γ .

(a) **Sum Rule**
$$\int_{\Gamma} (f(z) + g(z)) dz = \int_{\Gamma} f(z) dz + \int_{\Gamma} g(z) dz.$$

(b) **Multiple Rule**
$$\int_{\Gamma} \lambda f(z) dz = \lambda \int_{\Gamma} f(z) dz, \quad \text{where } \lambda \in \mathbb{C}.$$

9. Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then the **reverse path** of Γ , denoted by $\tilde{\Gamma}$, is the path with parametrisation $\tilde{\gamma}$, where

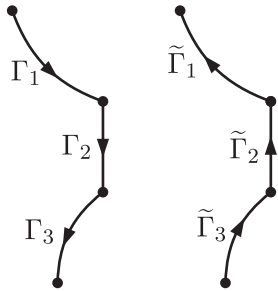
$$\tilde{\gamma}(t) = \gamma(a + b - t) \quad (t \in [a, b]).$$

10. As sets, Γ and $\tilde{\Gamma}$ are the same.
11. Let $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$ be a contour. The **reverse contour** $\tilde{\Gamma}$ of Γ is

$$\tilde{\Gamma} = \tilde{\Gamma}_n + \tilde{\Gamma}_{n-1} + \cdots + \tilde{\Gamma}_1.$$

12. **Reverse Contour Theorem** Let Γ be a contour, and let f be a function that is continuous on Γ . Then the integral of f along the reverse contour $\tilde{\Gamma}$ of Γ satisfies

$$\int_{\tilde{\Gamma}} f(z) dz = - \int_{\Gamma} f(z) dz.$$



Section 3 Evaluating contour integrals

1. Let f and F be functions defined on a region \mathcal{R} . Then F is a **primitive of f on \mathcal{R}** if F is analytic on \mathcal{R} and

$$F'(z) = f(z), \quad \text{for all } z \in \mathcal{R}.$$

2. **Fundamental Theorem of Calculus** Let f be a function that is continuous and has a primitive F on a region \mathcal{R} , and let Γ be a contour in \mathcal{R} with initial point α and final point β . Then

$$\int_{\Gamma} f(z) dz = F(\beta) - F(\alpha) = [F(z)]_{\alpha}^{\beta}.$$

3. **Contour Independence Theorem** Let f be a function that is continuous and has a primitive F on a region \mathcal{R} , and let Γ_1 and Γ_2 be contours in \mathcal{R} with the same initial point α and the same final point β . Then

$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

4. **Integration by Parts** Let f and g be functions that are analytic on a region \mathcal{R} , and suppose that f' and g' are continuous on \mathcal{R} . Let Γ be a contour in \mathcal{R} with initial point α and final point β . Then

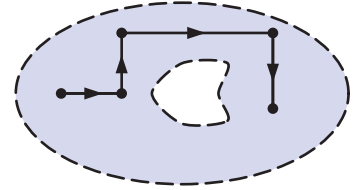
$$\int_{\Gamma} f(z)g'(z) dz = [f(z)g(z)]_{\alpha}^{\beta} - \int_{\Gamma} f'(z)g(z) dz.$$

5. A path or contour Γ is **closed** if its initial and final points coincide.
6. If Γ is a closed contour, then the value of any contour integral along Γ does not depend on the choice of initial point of Γ .
7. **Closed Contour Theorem** Let f be a function that is continuous and has a primitive F on a region \mathcal{R} . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour Γ in \mathcal{R} .

8. A **grid path** is a contour each of whose constituent smooth paths is a line segment parallel to either the real axis or the imaginary axis.
9. **Grid Path Theorem** Any two points in a region \mathcal{R} can be joined by a grid path in \mathcal{R} .
10. **Zero Derivative Theorem** Let f be a function that is analytic on a region \mathcal{R} , and let $f'(z) = 0$, for all z in \mathcal{R} . Then f is constant on \mathcal{R} .



Section 4 Estimating contour integrals

1. Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a smooth path. Then the **length of the path** Γ is

$$L(\Gamma) = \int_a^b |\gamma'(t)| dt.$$

The **length of a contour** is the sum of the lengths of its constituent smooth paths.

2. The length of a smooth path $\Gamma : \gamma(t)$ ($t \in [a, b]$) is unchanged if γ is replaced by any other smooth parametrisation of Γ .

A smooth path and its reverse path have the same length.

The length of a contour is independent of the way that the contour is split into smooth paths.

3. **Estimation Theorem** Let f be a function that is continuous on a contour Γ of length L , with

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then

$$\left| \int_{\Gamma} f(z) dz \right| \leq ML.$$

4. Let $g: [a, b] \rightarrow \mathbb{C}$ be a continuous function. Then

$$\left| \int_a^b g(t) dt \right| \leq \int_a^b |g(t)| dt.$$

Unit B2 Cauchy's Theorem

Section 1 Cauchy's Theorem

1. A path $\Gamma : \gamma(t)$ ($t \in [a, b]$) is **simple** if γ is one-to-one on $[a, b]$.

A path $\Gamma : \gamma(t)$ ($t \in [a, b]$) is **simple-closed** if it is closed *and* γ is one-to-one on $[a, b]$.

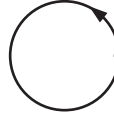
Since a contour is a special type of path, we also speak of **simple contours** and **simple-closed contours**.



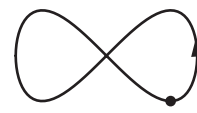
simple



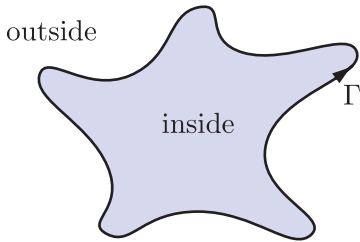
not simple



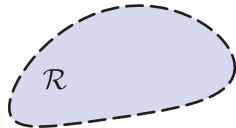
simple-closed



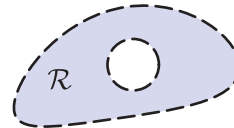
not simple-closed



2. **Jordan Curve Theorem** The complement $\mathbb{C} - \Gamma$ of a simple-closed path Γ is the union of two disjoint regions, one bounded and the other unbounded.
3. The bounded region in the complement of a simple-closed path Γ is called the **inside** of Γ and the unbounded region is called the **outside** of Γ .
4. A region \mathcal{R} is **simply connected** if, whenever Γ is a simple-closed path lying in \mathcal{R} , the inside of Γ also lies in \mathcal{R} .
5. To identify simply connected regions, we usually use the more informal definition that a region is simply connected if there are no holes in it.



simply connected



not simply connected

6. **Cauchy's Theorem** Let \mathcal{R} be a simply connected region, and let f be a function that is analytic on \mathcal{R} . Then

$$\int_{\Gamma} f(z) dz = 0,$$

for any closed contour Γ in \mathcal{R} .

7. **Contour Independence Theorem** Let \mathcal{R} be a simply connected region, let f be a function that is analytic on \mathcal{R} , and let Γ_1 and Γ_2 be contours in \mathcal{R} with the same initial point α and the same final point β . Then

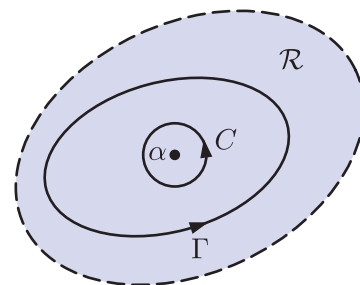
$$\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz.$$

8. **Convention** Unless otherwise specified, any simple-closed contour Γ appearing in a contour integral will be assumed to be traversed once anticlockwise, with the inside of Γ on the left.

9. **Shrinking Contour Theorem** Let \mathcal{R} be a simply connected region, let Γ be a simple-closed contour in \mathcal{R} , let α be a point inside Γ , and let f be a function that is analytic on $\mathcal{R} - \{\alpha\}$. Then

$$\int_{\Gamma} f(z) dz = \int_C f(z) dz,$$

where C is any circle lying inside Γ with centre α .



Section 2 Cauchy's Integral Formula

1. **Cauchy's Integral Formula** Let \mathcal{R} be a simply connected region, let Γ be a simple-closed contour in \mathcal{R} , and let f be a function that is analytic on \mathcal{R} . Then

$$f(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{z - \alpha} dz,$$

for any point α inside Γ .

2. **Liouville's Theorem** Every bounded entire function is a constant function.
3. **Fundamental Theorem of Algebra** Every non-constant polynomial function has at least one zero.
4. Any polynomial function p of degree $n \geq 1$ can be expressed in the form

$$p(z) = a(z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n),$$

where a is a non-zero complex number and $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ are all zeros of p , some of which may be repeated.

Section 3 Cauchy's Derivative Formulas

1. **Cauchy's First Derivative Formula** Let \mathcal{R} be a simply connected region, let Γ be a simple-closed contour in \mathcal{R} , and let f be a function that is analytic on \mathcal{R} . Then

$$f'(\alpha) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^2} dz,$$

for any point α inside Γ .

2. **Cauchy's n th Derivative Formula** Let \mathcal{R} be a simply connected region, let Γ be a simple-closed contour in \mathcal{R} , and let f be a function that is analytic on \mathcal{R} . Then, for any point α inside Γ , f is n -times differentiable at α and

$$f^{(n)}(\alpha) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n = 1, 2, \dots$$

3. **Analyticity of Derivatives** Let \mathcal{R} be a region, and let f be a function that is analytic on \mathcal{R} . Then f possesses derivatives of all orders on \mathcal{R} , so $f^{(1)}, f^{(2)}, f^{(3)}, \dots$ are all analytic on \mathcal{R} .

Section 4 Revision of contour integration

- Contour integrals can be evaluated by the following methods:
 - Parametrisation (using the definition of a contour integral) – items 2 and 6 in Section 2 of Unit B1
 - Closed Contour Theorem – item 7 in Section 3 of Unit B1
 - Cauchy's Theorem – item 6 in Section 1 of Unit B2
 - Cauchy's Integral Formula – item 1 in Section 2 of Unit B2
 - Cauchy's n th Derivative Formula – item 2 in Section 3 of Unit B2.
 See also Cauchy's Residue Theorem – item 1 in Section 2 of Unit C1.

- The partial fraction expansion of an expression $1/r(z)$, where

$$r(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_n)$$

and $\alpha_1, \alpha_2, \dots, \alpha_n$ are *distinct*, has the form

$$\frac{1}{r(z)} = \frac{A_1}{z - \alpha_1} + \frac{A_2}{z - \alpha_2} + \cdots + \frac{A_n}{z - \alpha_n}.$$

To determine the complex numbers A_1, A_2, \dots, A_n , multiply both sides by $r(z)$, and then equate coefficients of powers of z .

A factor $(z - \alpha)^m$ in $r(z)$ leads to a sum of m terms

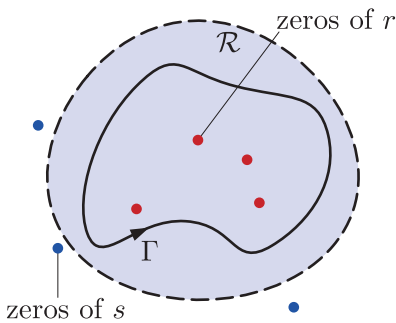
$$\frac{B_1}{z - \alpha} + \frac{B_2}{(z - \alpha)^2} + \cdots + \frac{B_m}{(z - \alpha)^m}$$

in the partial fraction expansion of $1/r(z)$.

- Strategy for evaluating contour integrals** To evaluate the integral $\int_{\Gamma} \frac{g(z)}{p(z)} dz$, where

- Γ is a simple-closed contour
- g is analytic on a simply connected region containing Γ
- p is a polynomial function with no zeros on Γ :

- (1) Factorise $p(z)$ as $r(z)s(z)$, where the zeros of r lie inside Γ and the zeros of s lie outside Γ . Then the function $f = g/s$ is analytic on a simply connected region \mathcal{R} that contains Γ but does not contain the zeros of s .
- (2) Expand $1/r(z)$ in partial fractions.
- (3) Expand $\int_{\Gamma} \frac{f(z)}{r(z)} dz$ as a sum of integrals that can be evaluated using Cauchy's Integral and n th Derivative Formulas.



Section 5 Proof of Cauchy's Theorem

- Primitive Theorem** Let f be a function that is analytic on a simply connected region \mathcal{R} . Then f has a primitive on \mathcal{R} .
- Morera's Theorem** Let f be a function that is continuous on a region \mathcal{R} and satisfies

$$\int_{\Gamma} f(z) dz = 0,$$

for all rectangular contours Γ in \mathcal{R} . Then f is analytic on \mathcal{R} .

Unit B3 Taylor series

Section 1 Complex series

- Given a sequence (z_n) of complex numbers, the expression

$$z_1 + z_2 + z_3 + \cdots$$

is called an **infinite series**, or simply a **series**. The number z_n is called the **n th term** of the series.

The **n th partial sum** of the series is the complex number

$$s_n = z_1 + z_2 + \cdots + z_n = \sum_{k=1}^n z_k.$$

- We sometimes describe series as **complex series**. A **real series** is a series with terms that are all real numbers.
- The complex series $z_1 + z_2 + z_3 + \cdots$ is **convergent** with **sum** s if the sequence (s_n) of partial sums converges to s . In this case we say that the series **converges** to s , and write

$$z_1 + z_2 + z_3 + \cdots = s \quad \text{or} \quad \sum_{n=1}^{\infty} z_n = s.$$

The series is **divergent**, and we say that it **diverges**, if the sequence (s_n) diverges.

- Theorem** If $\sum_{n=1}^{\infty} z_n$ converges, then (z_n) is a null sequence.
- Non-null Test** If the sequence (z_n) is not null, then the series $\sum_{n=1}^{\infty} z_n$ diverges.
- The converse of the Non-null Test is *false*, because if (z_n) is a null sequence, then it does *not* follow that $\sum_{n=1}^{\infty} z_n$ converges: it may converge or it may diverge.
- The series $\sum_{n=0}^{\infty} az^n = a + az + az^2 + \cdots$, where $a, z \in \mathbb{C}$, is called a **geometric series** with **common ratio** z .
- Geometric series** Consider the series $\sum_{n=0}^{\infty} az^n$, where $a, z \in \mathbb{C}$.
 - If $|z| < 1$, then the series converges to $a/(1 - z)$.
 - If $|z| \geq 1$ and $a \neq 0$, then the series diverges.
- Theorem** The series $\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ converges if $p > 1$ and diverges if $p \leq 1$.

- When $p = 1$ we obtain the (divergent) **harmonic series**

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots.$$

11. **Combination Rules for Series** If the series $\sum_{n=1}^{\infty} z_n$ and $\sum_{n=1}^{\infty} w_n$ both converge, then
 - (a) **Sum Rule** $\sum_{n=1}^{\infty} (z_n + w_n) = \sum_{n=1}^{\infty} z_n + \sum_{n=1}^{\infty} w_n$
 - (b) **Multiple Rule** $\sum_{n=1}^{\infty} \lambda z_n = \lambda \sum_{n=1}^{\infty} z_n$, for $\lambda \in \mathbb{C}$.
12. **Theorem** The series $\sum_{n=1}^{\infty} z_n$ converges if and only if both $\sum_{n=1}^{\infty} \operatorname{Re} z_n$ and $\sum_{n=1}^{\infty} \operatorname{Im} z_n$ converge. In this case

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} \operatorname{Re} z_n + i \sum_{n=1}^{\infty} \operatorname{Im} z_n.$$
13. **Comparison Test** If $\sum_{n=1}^{\infty} a_n$ is a convergent real series of non-negative terms, and

$$|z_n| \leq a_n, \quad \text{for } n = 1, 2, \dots,$$
 then the series $\sum_{n=1}^{\infty} z_n$ converges.
14. The complex series $\sum_{n=1}^{\infty} z_n$ is **absolutely convergent** (or **converges absolutely**) if the real series $\sum_{n=1}^{\infty} |z_n|$ is convergent.
15. **Absolute Convergence Test** If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then the series $\sum_{n=1}^{\infty} z_n$ converges.
16. The converse of the Absolute Convergence Test is *false*, because there are convergent series that do *not* converge absolutely.
17. **Triangle Inequality for Series** If the series $\sum_{n=1}^{\infty} z_n$ is absolutely convergent, then

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|.$$
18. **Ratio Test** Suppose that $\sum_{n=1}^{\infty} z_n$ is a complex series for which

$$\left| \frac{z_{n+1}}{z_n} \right| \rightarrow l \text{ as } n \rightarrow \infty.$$
 - (a) If $0 \leq l < 1$, then $\sum_{n=1}^{\infty} z_n$ converges absolutely (so it converges).
 - (b) If $l > 1$, then $\sum_{n=1}^{\infty} z_n$ diverges.

19. The Ratio Test yields no information if $l = 1$. The case $l > 1$ includes the situation where $\left| \frac{z_{n+1}}{z_n} \right| \rightarrow \infty$ as $n \rightarrow \infty$.

Section 2 Power series

1. An expression of the form

$$\sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \cdots,$$

where z is a complex variable and $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$, is called a **power series about 0**.

More generally, if $\alpha \in \mathbb{C}$, then an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n = a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2 + \cdots,$$

where $a_n \in \mathbb{C}$, $n = 0, 1, 2, \dots$, is called a **power series about α** .

2. A power series **converges on a set S** if, for each $z \in S$, the corresponding series converges.

3. Let $A = \left\{ z : \sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ converges} \right\}$. The function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n \quad (z \in A)$$

is called the **sum function** of the power series.

4. **Radius of Convergence Theorem** For a given power series

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n = a_0 + a_1 (z - \alpha) + a_2 (z - \alpha)^2 + \cdots,$$

precisely one of the following possibilities occurs:

- (a) the series converges only for $z = \alpha$
- (b) the series converges for all z
- (c) there is a number $R > 0$ such that

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ converges (absolutely) if } |z - \alpha| < R,$$

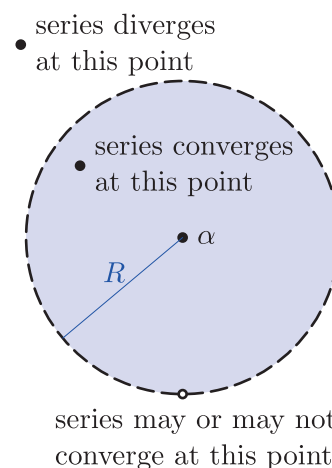
and

$$\sum_{n=0}^{\infty} a_n (z - \alpha)^n \text{ diverges if } |z - \alpha| > R.$$

5. The **radius of convergence** of a power series satisfying case (c) from the Radius of Convergence Theorem is the number R .

We extend this definition of the radius of convergence R by writing $R = 0$ for case (a), and $R = \infty$ for case (b).

6. All the convergence tests in Section 1 of Unit B3 can be applied to power series, since, for each value of z , a power series is just a series.



7. Radius of Convergence Formula The power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n$$

has radius of convergence

$$R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|,$$

provided that this limit exists (or is ∞).

8. Let R be the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n.$$

Then the **disc of convergence** of the power series is the open disc $\{z : |z - \alpha| < R\}$. The disc of convergence is interpreted to be the empty set \emptyset if $R = 0$, and to be \mathbb{C} if $R = \infty$.

9. A power series may converge at none, some, or all of the points on the boundary of its disc of convergence.

10. Differentiation Rule for Power Series The power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \quad \text{and} \quad \sum_{n=1}^{\infty} n a_n(z - \alpha)^{n-1}$$

have the same radius of convergence R . Furthermore, if

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

then f is analytic on the disc of convergence $\{z : |z - \alpha| < R\}$, and

$$f'(z) = \sum_{n=1}^{\infty} n a_n(z - \alpha)^{n-1}, \quad \text{for } |z - \alpha| < R.$$

11. Integration Rule for Power Series The power series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n \quad \text{and} \quad \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - \alpha)^{n+1}$$

have the same radius of convergence R .

Furthermore, if $f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n$, then the function

$$F(z) = b_0 + \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - \alpha)^{n+1}, \quad \text{where } b_0 \text{ is any constant,}$$

is a primitive of f on $\{z : |z - \alpha| < R\}$.

Section 3 Taylor's Theorem

1. **Taylor's Theorem** Let f be a function that is analytic on the open disc $D = \{z : |z - \alpha| < r\}$. Then

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n, \quad \text{for } z \in D.$$

Moreover, this representation of f is unique, in the sense that if

$$f(z) = \sum_{n=0}^{\infty} a_n (z - \alpha)^n, \quad \text{for } z \in D,$$

then $a_n = f^{(n)}(\alpha)/n!$, for $n = 0, 1, 2, \dots$.

2. In Taylor's Theorem, the term $f^{(n)}(\alpha)/n!$ makes sense for $n = 0$ because, by convention, we take $0! = 1$ and $f^{(0)}(z) = f(z)$.
3. Let f be a function with derivatives $f^{(1)}(\alpha), f^{(2)}(\alpha), f^{(3)}(\alpha), \dots$ at the point α . Then the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!} (z - \alpha)^n$$

is called the **Taylor series about α for f** . The coefficient $f^{(n)}(\alpha)/n!$ is known as the **n th Taylor coefficient of f at α** .

4. The n th Taylor coefficient of f at α can be written as

$$\frac{f^{(n)}(\alpha)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz,$$

where C is a circle centred at α that lies in the open disc $D = \{z : |z - \alpha| < r\}$ on which f is analytic.

5. Let f be an entire function. Then, for any point α , the Taylor series about α for f converges to $f(z)$ for each $z \in \mathbb{C}$.
6. Let A be a set for which $z \in A$ if and only if $-z \in A$.

A function $f: A \rightarrow \mathbb{C}$ is an **even function** if

$$f(-z) = f(z), \quad \text{for } z \in A,$$

and f is an **odd function** if

$$f(-z) = -f(z), \quad \text{for } z \in A.$$

7. **Theorem** Let f be a function that is analytic at 0 with Taylor series about 0 given by

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

- (a) If f is an even function, then $a_n = 0$ for n odd.
- (b) If f is an odd function, then $a_n = 0$ for n even.

Thus if f is even, then its Taylor series about 0 has only even powers, and if f is odd, then its Taylor series about 0 has only odd powers.

8. Basic Taylor series

$$(1 - z)^{-1} = 1 + z + z^2 + z^3 + \cdots, \quad \text{for } |z| < 1$$

$$\exp z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\operatorname{Log}(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots, \quad \text{for } |z| < 1$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\sinh z = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \frac{z^7}{7!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

$$\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots, \quad \text{for } z \in \mathbb{C}$$

9. **Binomial Series** Let $\alpha \in \mathbb{C}$. The **binomial series** about 0 for the function $f(z) = (1 + z)^\alpha$ is

$$(1 + z)^\alpha = 1 + \binom{\alpha}{1}z + \binom{\alpha}{2}z^2 + \binom{\alpha}{3}z^3 + \cdots, \quad \text{for } |z| < 1,$$

where

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)(\alpha - 2) \cdots (\alpha - (n - 1))}{n!}.$$

If α is a positive integer or zero, then the binomial series reduces to a polynomial, so it converges for all $z \in \mathbb{C}$; otherwise, the series is a power series whose radius of convergence is 1.

10. The coefficients $\binom{\alpha}{n}$ are called the **binomial coefficients** of the binomial series. (See item 11 in Section 1 of Unit A1 for binomial coefficients using positive integers.)

Section 4 Manipulating Taylor series

1. **Combination Rules for Power Series** Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R'.$$

- (a) **Sum Rule** Let $r = \min\{R, R'\}$. Then

$$(f + g)(z) = \sum_{n=0}^{\infty} (a_n + b_n)(z - \alpha)^n, \quad \text{for } |z - \alpha| < r.$$

- (b) **Multiple Rule** If $\lambda \in \mathbb{C}$, then

$$(\lambda f)(z) = \sum_{n=0}^{\infty} \lambda a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R.$$

2. Product Rule for Power Series

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(z) = \sum_{n=0}^{\infty} b_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R'.$$

Let $r = \min\{R, R'\}$. Then

$$(fg)(z) = \sum_{n=0}^{\infty} c_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each positive integer n ,

$$c_n = a_0b_n + a_1b_{n-1} + a_2b_{n-2} + \cdots + a_nb_0.$$

3. Substitution Rules for Power Series

The substitution

$$w = \lambda z^k, \quad \text{where } \lambda \neq 0, k \in \mathbb{N},$$

changes a power series in powers of w with radius of convergence R to a power series in powers of z with radius of convergence $\sqrt[k]{R/|\lambda|}$.

The substitution

$$w = z + \beta - \alpha$$

changes a power series in powers of $w - \beta$ to a power series in powers of $z - \alpha$, and preserves the radius of convergence.

4. Composition Rule for Power Series

Let

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < R,$$

$$g(w) = \sum_{n=0}^{\infty} b_n(w - \beta)^n, \quad \text{for } |w - \beta| < R'.$$

If $\beta = f(\alpha)$, then, for some $r > 0$,

$$g(f(z)) = \sum_{n=0}^{\infty} c_n(z - \alpha)^n, \quad \text{for } |z - \alpha| < r,$$

where, for each n , the number c_n is the coefficient of $(z - \alpha)^n$ in

$$\sum_{k=0}^n b_k \left(\sum_{l=1}^n a_l (z - \alpha)^l \right)^k.$$

- Make sure that you check the condition $\beta = f(\alpha)$ when applying the Composition Rule for Power Series.

- Theorem** Let f be a function that is analytic and unbounded on the open disc $D = \{z : |z - \alpha| < R\}$ centred at α of radius R .

Then D is the disc of convergence for the Taylor series about α for f , so this Taylor series has radius of convergence R .

Section 5 The Uniqueness Theorem

1. Let f be a function that is analytic at α . If

$$f(\alpha) = f^{(1)}(\alpha) = f^{(2)}(\alpha) = \cdots = f^{(k-1)}(\alpha) = 0, \text{ but } f^{(k)}(\alpha) \neq 0,$$

then f has a **zero of (finite) order k at α** .

A zero of order 1 is called a **simple zero**.

2. **Theorem** A function f is analytic at a point α , and has a zero of order k at α , if and only if, for some $r > 0$,

$$f(z) = (z - \alpha)^k g(z), \quad \text{for } |z - \alpha| < r,$$

where g is a function that is analytic at α , and $g(\alpha) \neq 0$.

3. **Theorem** Let f be a function that is analytic on a region \mathcal{R} and not identically zero on \mathcal{R} . Then any zero of f is of finite order.
4. A zero α of a function f is said to be **isolated** if there is a disc centred at α that contains no other zeros of f .
5. **Isolated zeros** A zero of finite order is isolated.
6. **Theorem** Let f be a function that is analytic on a region \mathcal{R} , and let S be a set of zeros of f in \mathcal{R} that has a limit point in \mathcal{R} . Then f is identically zero on \mathcal{R} .
7. We say that two functions f and g **agree** on a set S if $f(z) = g(z)$, for all $z \in S$.
8. **Uniqueness Theorem** Let f and g be functions that are analytic on a region \mathcal{R} , and suppose that f and g agree on a subset S of \mathcal{R} , where S has a limit point in \mathcal{R} . Then f and g agree throughout \mathcal{R} .

Unit B4 Laurent series

Section 1 Singularities

1. A function f has an **isolated singularity** or, more briefly, a **singularity**, at the point α if f is analytic on a punctured open disc $\{z : 0 < |z - \alpha| < r\}$, where $r > 0$, but not at α itself.
2. Let f be a function with domain A , and suppose that α is a limit point of A . The function f **tends to infinity as z tends to α** if, for each sequence (z_n) in $A - \{\alpha\}$ such that $z_n \rightarrow \alpha$, we have

$$f(z_n) \rightarrow \infty.$$

(Or, equivalently, for each $M > 0$, there is a $\delta > 0$ such that

$$|f(z)| > M, \quad \text{for all } z \in A \text{ with } 0 < |z - \alpha| < \delta.)$$

We write $f(z) \rightarrow \infty$ as $z \rightarrow \alpha$.

3. **Reciprocal Rule for Functions** Let f be a function with domain A , and suppose that α is a limit point of A . Then

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha \quad \Longleftrightarrow \quad \lim_{z \rightarrow \alpha} \frac{1}{f(z)} = 0.$$

4. $\lim_{z \rightarrow 0} \frac{\sin z}{z} = 1.$
5. Let f be a function that has a singularity at the point α . Then f has a **removable singularity** at α if there is a function g that is analytic on an open disc $\{z : |z - \alpha| < r\}$ such that

$$f(z) = g(z), \quad \text{for } 0 < |z - \alpha| < r.$$

The function g is called an **analytic extension** of f to $\{z : |z - \alpha| < r\}$.

6. Let f be a function that has a singularity at the point α . Then f has a **simple pole** at α if there is a function g that is analytic on an open disc $\{z : |z - \alpha| < r\}$ such that $g(\alpha) \neq 0$ and

$$f(z) = \frac{g(z)}{z - \alpha}, \quad \text{for } 0 < |z - \alpha| < r.$$

7. Let f be a function that has a singularity at the point α . Then f has a **pole of order k** at α if there is a function g that is analytic on an open disc $\{z : |z - \alpha| < r\}$ such that $g(\alpha) \neq 0$ and

$$f(z) = \frac{g(z)}{(z - \alpha)^k}, \quad \text{for } 0 < |z - \alpha| < r.$$

8. Let f be a function that has a singularity at the point α . Then f has an **essential singularity** at α if the singularity at α is neither a removable singularity nor a pole.
9. **Theorem** Let f be a function that has a singularity at the point α . If $f(z)$ does not tend to a finite limit or to ∞ as z tends to α , then f has an essential singularity at α .

Section 2 Laurent's Theorem

1. An expression of the form

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n = \cdots + \frac{a_{-2}}{(z - \alpha)^2} + \frac{a_{-1}}{(z - \alpha)} + a_0 + a_1(z - \alpha) + \cdots,$$

where z is a complex variable, $\alpha \in \mathbb{C}$ and $a_n \in \mathbb{C}$, for $n \in \mathbb{Z}$, is called an **extended power series about α** .

For a given z , the extended power series **converges** if the series

$$\sum_{n=0}^{\infty} a_n(z - \alpha)^n = a_0 + a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots,$$

$$\sum_{n=1}^{\infty} a_{-n}(z - \alpha)^{-n} = \frac{a_{-1}}{(z - \alpha)} + \frac{a_{-2}}{(z - \alpha)^2} + \cdots$$

both converge.

These two series are called the **analytic part** and **singular part** of the extended power series, respectively.

2. At a point z for which the extended power series converges, we can form the **sum** of the extended power series at z by adding the sums of the analytic and singular parts:

$$\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n = \sum_{n=0}^{\infty} a_n(z - \alpha)^n + \sum_{n=1}^{\infty} a_{-n}(z - \alpha)^{-n}.$$

3. Let $A = \left\{ z : \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \text{ converges} \right\}$. The function

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n \quad (z \in A)$$

is called the **sum function** of the extended power series.

4. If the analytic part of an extended power series has disc of convergence $\{z : |z - \alpha| < s\}$ and the singular part converges on $\{z : |z - \alpha| > r\}$, then the extended power series has **annulus of convergence**

$$\begin{aligned} A &= \{z : |z - \alpha| < s\} \cap \{z : |z - \alpha| > r\} \\ &= \{z : r < |z - \alpha| < s\}. \end{aligned}$$

This set may take any one of the following forms:

- an open annulus $(0 < r < s < \infty)$
- a punctured open disc $(r = 0 < s < \infty)$
- a punctured plane $(r = 0, s = \infty)$
- the outside of a closed disc $(0 < r < s = \infty)$
- the empty set $(r \geq s)$.

5. **Laurent's Theorem** Let f be a function that is analytic on the open annulus

$$A = \{z : r < |z - \alpha| < s\}, \quad \text{where } 0 \leq r < s \leq \infty.$$

Then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A,$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \alpha)^{n+1}} dz, \quad \text{for } n \in \mathbb{Z},$$

and C is any circle lying in A with centre α .

Moreover, this representation of f on A as an extended power series about α is unique.

6. The representation

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n, \quad \text{for } z \in A,$$

determined by Laurent's Theorem is called the **Laurent series about α for the function f on the annulus A .**

7. If A is a punctured open disc, then the representation is called the **Laurent series about α for the function f .**
8. A function f may have different Laurent series about α on different annuli.
9. A Laurent series may converge at none, some, or all of the points on the boundary of its annulus of convergence.
10. **Theorem** Let f be a function that has a singularity at the point α , and suppose that the Laurent series about α for f is

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n.$$

Then

- (a) f has a removable singularity at α if and only if

$$a_n = 0 \text{ for all } n < 0$$

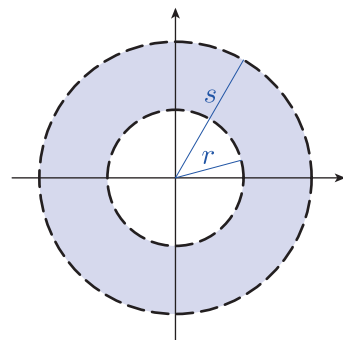
- (b) f has a pole of order $k \in \mathbb{N}$ at α if and only if

$$a_{-k} \neq 0 \text{ and } a_n = 0 \text{ for all } n < -k$$

- (c) f has an essential singularity at α if and only if

$$a_n \neq 0 \text{ for infinitely many } n < 0.$$

11. Laurent series can be obtained from known Taylor series by using substitutions and by using partial fractions.



Section 3 Behaviour near a singularity

1. **Theorem** Let f be a function that has a singularity at the point α . Then the following statements are equivalent:
 - (a) f has a removable singularity at α
 - (b) $\lim_{z \rightarrow \alpha} f(z)$ exists
 - (c) f is bounded on $\{z : 0 < |z - \alpha| < r\}$, for some $r > 0$
 - (d) $\lim_{z \rightarrow \alpha} (z - \alpha)f(z) = 0$.
2. **Theorem** Let f be a function that has a singularity at the point α , and let $k \in \mathbb{N}$. Then the following statements are equivalent:
 - (a) f has a pole of order k at α
 - (b) $\lim_{z \rightarrow \alpha} (z - \alpha)^k f(z)$ exists, and is non-zero
 - (c) $1/f$ has a removable singularity at α which, when removed, gives rise to a zero of order k at α .
3. Let f be a function that has a singularity at the point α . Then f has a pole at α if and only if

$$f(z) \rightarrow \infty \text{ as } z \rightarrow \alpha.$$
4. **Casorati–Weierstrass Theorem** Suppose that a function f has an essential singularity at α . Let D be any punctured open disc $\{z : 0 < |z - \alpha| < \delta\}$ centred at α , and let w be any complex number. Then, for any positive number ε ,

$$\text{there exists } z \in D \text{ such that } |f(z) - w| < \varepsilon.$$

Section 4 Evaluating integrals using Laurent series

1. Let f be a function that is analytic on the punctured disc $D = \{z : 0 < |z - \alpha| < r\}$. Then

$$\int_C \frac{f(w)}{(w - \alpha)^{n+1}} dw = 2\pi i a_n,$$
 where $\sum_{n=-\infty}^{\infty} a_n(z - \alpha)^n$ is the Laurent series about α for f , and C is any circle lying in D with centre α .
2. Let f be a function that is analytic on a punctured disc with centre α . The **residue of f at α** is the coefficient a_{-1} of $(z - \alpha)^{-1}$ in the Laurent series about α for f . It is denoted by $\text{Res}(f, \alpha)$.
3. By item 1 with $n = -1$,

$$\int_C f(z) dz = 2\pi i \text{Res}(f, \alpha),$$
 where C is any circle lying in $D = \{z : 0 < |z - \alpha| < r\}$ with centre α .

Unit C1 Residues

Section 1 Calculating residues

- Theorem** Let f be a function that has singularities at the points α and $-\alpha$.
 - If f is an odd function, then $\text{Res}(f, -\alpha) = \text{Res}(f, \alpha)$.
 - If f is an even function, then $\text{Res}(f, -\alpha) = -\text{Res}(f, \alpha)$.
- Theorem** Let f be a function that has a singularity at the point α , and suppose that the limit $\lim_{z \rightarrow \alpha} (z - \alpha)f(z)$ exists. Then

$$\text{Res}(f, \alpha) = \lim_{z \rightarrow \alpha} (z - \alpha)f(z).$$

Furthermore, f has a simple pole at α if the limit is non-zero, and it has a removable singularity at α if the limit is 0.

- g/h Rule** Let $f(z) = g(z)/h(z)$, where g and h are functions that are analytic at the point α , and $h(\alpha) = 0$ and $h'(\alpha) \neq 0$. Then

$$\text{Res}(f, \alpha) = g(\alpha)/h'(\alpha).$$

- Cover-up Rule** Let $f(z) = \frac{g(z)}{z - \alpha}$, where g is a function that is analytic at α . Then

$$\text{Res}(f, \alpha) = g(\alpha).$$

- When applying the Cover-up Rule, make sure that you cover up only a factor of the form $z - \alpha$.
- The g/h Rule and the Cover-up Rule can be used to calculate residues only at singularities that are simple poles or removable singularities.
- Theorem** Let f be a function that has a pole of order k at the point α . Then

$$\text{Res}(f, \alpha) = \frac{1}{(k-1)!} \lim_{z \rightarrow \alpha} \left(\frac{d^{k-1}}{dz^{k-1}} \left((z - \alpha)^k f(z) \right) \right).$$

Section 2 The Residue Theorem

- Cauchy's Residue Theorem** Let \mathcal{R} be a simply connected region, and let f be a function that is analytic on \mathcal{R} except for a finite number of singularities. Let Γ be any simple-closed contour in \mathcal{R} , not passing through any of these singularities. Then

$$\int_{\Gamma} f(z) dz = 2\pi i S,$$

where S is the sum of the residues of f at those singularities that lie inside Γ .

2. Strategy for evaluating real trigonometric integrals

To evaluate a real integral of the form

$$\int_0^{2\pi} \Phi(\cos t, \sin t) dt,$$

where Φ is a function of two real variables, proceed as follows.

(1) Replace

$$\cos t \text{ by } \frac{1}{2}(z + z^{-1}), \quad \sin t \text{ by } \frac{1}{2i}(z - z^{-1}), \quad dt \text{ by } \frac{1}{iz} dz,$$

to obtain a contour integral of the form $\int_C f(z) dz$ around the unit circle $C = \{z : |z| = 1\}$. In order for the strategy to apply, the function f must be analytic with finitely many singularities on a simply connected region that contains C , and none of the singularities can lie on C .

(2) Locate the singularities of the function f lying inside C , and calculate the residues of f at these points.

(3) Evaluate the given integral by calculating

$$2\pi i \times (\text{the sum of the residues found in step 2}).$$

Section 3 Evaluating improper integrals

- Let f be a function defined on an unbounded interval (a, ∞) , and suppose that $\alpha \in \mathbb{C}$. The function f has **limit α as r tends to ∞** if for each real sequence (r_n) in (a, ∞) such that $r_n \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$f(r_n) \rightarrow \alpha \text{ as } n \rightarrow \infty.$$

(Or, equivalently, for each $\varepsilon > 0$, there is an integer N such that

$$|f(r) - \alpha| < \varepsilon, \quad \text{for all } r > N.)$$

We write either

$$\lim_{r \rightarrow \infty} f(r) = \alpha \quad \text{or} \quad f(r) \rightarrow \alpha \text{ as } r \rightarrow \infty.$$

- Combination Rules for Limits of Functions** Let f and g be functions such that

$$\lim_{r \rightarrow \infty} f(r) = \alpha \quad \text{and} \quad \lim_{r \rightarrow \infty} g(r) = \beta.$$

- Sum Rule** $\lim_{r \rightarrow \infty} (f(r) + g(r)) = \alpha + \beta.$
 - Multiple Rule** $\lim_{r \rightarrow \infty} (\lambda f(r)) = \lambda\alpha, \quad \text{for } \lambda \in \mathbb{C}.$
 - Product Rule** $\lim_{r \rightarrow \infty} (f(r)g(r)) = \alpha\beta.$
 - Quotient Rule** $\lim_{r \rightarrow \infty} (f(r)/g(r)) = \alpha/\beta, \quad \text{provided that } \beta \neq 0.$
- If p and q are polynomial functions such that the degree of q exceeds the degree of p , then

$$\lim_{r \rightarrow \infty} \frac{p(r)}{q(r)} = 0.$$

4. Let f be a function that is continuous on \mathbb{R} . Then the **improper**

integral $\int_{-\infty}^{\infty} f(t) dt$ is

$$\int_{-\infty}^{\infty} f(t) dt = \lim_{r \rightarrow \infty} \int_{-r}^r f(t) dt,$$

provided that this limit exists.

Let f be a function that is continuous on the interval $[a, \infty)$. Then the

improper integral $\int_a^{\infty} f(t) dt$ is

$$\int_a^{\infty} f(t) dt = \lim_{r \rightarrow \infty} \int_a^r f(t) dt,$$

provided that this limit exists.

5. **Theorem** Let f be a function that is continuous on \mathbb{R} .

(a) If f is an odd function, then

$$\int_{-\infty}^{\infty} f(t) dt = 0.$$

(b) If f is an even function, then

$$\int_{-\infty}^{\infty} f(t) dt = 2 \int_0^{\infty} f(t) dt,$$

provided that these improper integrals exist.

6. Let f be a function that is continuous at all points of an interval $[a, b]$ except the point $c \in (a, b)$, at which f may or may not be defined.

Then the **improper integral** $\int_a^b f(t) dt$ is

$$\int_a^b f(t) dt = \lim_{\varepsilon \rightarrow 0} \left(\int_a^{c-\varepsilon} f(t) dt + \int_{c+\varepsilon}^b f(t) dt \right),$$

provided that this limit, which is taken through *positive* values of ε , exists.

Let f be a function that is continuous at all points of \mathbb{R} except the

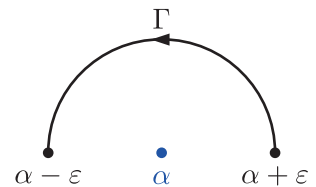
point c . Then the **improper integral** $\int_{-\infty}^{\infty} f(t) dt$ is

$$\begin{aligned} \int_{-\infty}^{\infty} f(t) dt &= \lim_{r \rightarrow \infty} \int_{-r}^r f(t) dt \\ &= \lim_{r \rightarrow \infty} \left(\lim_{\varepsilon \rightarrow 0} \left(\int_{-r}^{c-\varepsilon} f(t) dt + \int_{c+\varepsilon}^r f(t) dt \right) \right), \end{aligned}$$

provided that these limits exist.

7. **Round-the-Pole Lemma** Suppose that f is a function that is analytic on a punctured disc $\{z : 0 < |z - \alpha| < \delta\}$ and has a simple pole at α . Let Γ be the upper half of the circle centred at α of radius ε , where $\varepsilon < \delta$, traversed from $\alpha + \varepsilon$ to $\alpha - \varepsilon$. Then

$$\lim_{\varepsilon \rightarrow 0} \int_{\Gamma} f(z) dz = \pi i \operatorname{Res}(f, \alpha).$$



8. **Theorem** Let p and q be polynomial functions such that

- the degree of q exceeds that of p by at least two
- any poles of p/q on the real axis are simple.

Then

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function p/q at the poles in the upper half-plane, and T is the sum of the residues of p/q at the poles on the real axis.

9. **Theorem** Let p and q be polynomial functions such that

- the degree of q exceeds that of p by at least one
- any poles of p/q on the real axis are simple.

Then, if $k > 0$,

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt} dt = 2\pi i S + \pi i T,$$

where S is the sum of the residues of the function

$$f(z) = \frac{p(z)}{q(z)} e^{ikz}$$

at the poles in the upper half-plane, and T is the sum of the residues of f at the poles on the real axis.

10. If p and q are real polynomial functions, then we can equate the real and imaginary parts of the equation

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} e^{ikt} dt = 2\pi i S + \pi i T$$

to obtain the values of the real improper integrals

$$\int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \cos kt dt \quad \text{and} \quad \int_{-\infty}^{\infty} \frac{p(t)}{q(t)} \sin kt dt.$$

11. **Jordan's Lemma** Let Γ be the upper half of the circle centred at 0 of radius r , traversed from r to $-r$, and suppose that f is a function that is continuous on Γ and satisfies

$$|f(z)| \leq M, \quad \text{for } z \in \Gamma.$$

Then, for $k > 0$, we have

$$\left| \int_{\Gamma} f(z) e^{ikz} dz \right| \leq \frac{M\pi}{k}.$$

Section 4 Summing series

1. **Theorem** Let h be an even function that is analytic on \mathbb{C} except for poles at the points $\alpha_1, \alpha_2, \dots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$. Suppose that the function $f(z) = (\pi \cot \pi z)h(z)$ is such that

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0.$$

Then

$$\sum_{n=1}^{\infty} h(n) = -\frac{1}{2} \left(\text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right).$$

2. If $f(z) = (\pi \cot \pi z)h(z)$, where h is analytic at 0, then

$$\text{Res}(f, 0) = h(0).$$

3. For each $N = 1, 2, \dots$,

$$|\cot \pi z| \leq 2, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$.

4. **Theorem** Let h be an even function that is analytic on \mathbb{C} except for poles at the points $\alpha_1, \alpha_2, \dots, \alpha_k$ (none of which is an integer), and possibly at 0, and let S_N be the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$. Suppose that the function $f(z) = (\pi \operatorname{cosec} \pi z)h(z)$ is such that

$$\lim_{N \rightarrow \infty} \int_{S_N} f(z) dz = 0.$$

Then

$$\sum_{n=1}^{\infty} (-1)^n h(n) = -\frac{1}{2} \left(\text{Res}(f, 0) + \sum_{j=1}^k \text{Res}(f, \alpha_j) \right).$$

5. If $f(z) = (\pi \operatorname{cosec} \pi z)h(z)$, where h is analytic at 0, then

$$\text{Res}(f, 0) = h(0).$$

6. For each $N = 1, 2, \dots$,

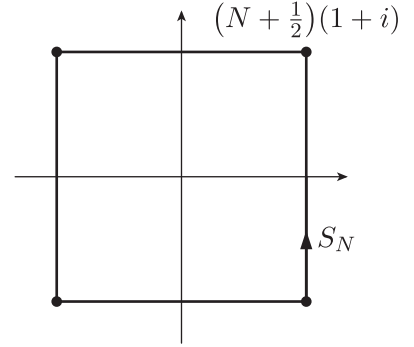
$$|\operatorname{cosec} \pi z| \leq 1, \quad \text{for } z \in S_N,$$

where S_N is the square contour with vertices at $(N + \frac{1}{2})(\pm 1 \pm i)$.

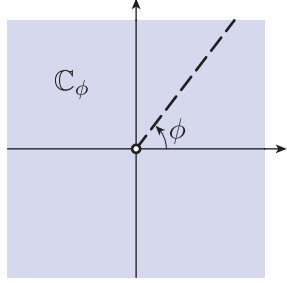
7. The Laurent series about 0 for \cot and cosec are

$$\cot z = \frac{1}{z} - \frac{1}{3}z - \frac{1}{45}z^3 - \dots,$$

$$\operatorname{cosec} z = \frac{1}{z} + \frac{1}{6}z + \frac{7}{360}z^3 + \dots.$$



Section 5 Analytic continuation



1. For $\phi \in \mathbb{R}$, the function Arg_ϕ is defined by

$$\text{Arg}_\phi(z) = \theta \quad (z \in \mathbb{C} - \{0\}),$$

where θ is the argument of z lying in the interval $(\phi - 2\pi, \phi]$.

2. The function Arg_π is just the principal argument function Arg .
3. For $\phi \in \mathbb{R}$, the cut plane \mathbb{C}_ϕ is defined by

$$\mathbb{C}_\phi = \{re^{i\theta} : r > 0, \phi - 2\pi < \theta < \phi\}.$$

4. **Theorem** For all $\phi \in \mathbb{R}$, Arg_ϕ is continuous on \mathbb{C}_ϕ .

5. For $\phi \in \mathbb{R}$, the function Log_ϕ is defined by

$$\text{Log}_\phi(z) = \log |z| + i \text{Arg}_\phi(z) \quad (z \in \mathbb{C} - \{0\}).$$

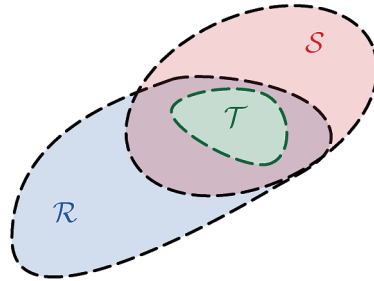
6. **Theorem** For all $\phi \in \mathbb{R}$, the function Log_ϕ is analytic on \mathbb{C}_ϕ with derivative

$$\text{Log}'_\phi(z) = \frac{1}{z} \quad (z \in \mathbb{C}_\phi).$$

7. Let f and g be analytic functions whose domains are the regions \mathcal{R} and \mathcal{S} , respectively. Then f and g are **direct analytic continuations** of each other if there is a region $\mathcal{T} \subseteq \mathcal{R} \cap \mathcal{S}$ such that

$$f(z) = g(z), \quad \text{for } z \in \mathcal{T}.$$

We also say that g is a **direct analytic continuation** of f from \mathcal{R} to \mathcal{S} , and vice versa.



8. Let f be a function that is continuous on the interval $(0, \infty)$. Then the **improper integral** $\int_0^\infty f(t) dt$ is

$$\int_0^\infty f(t) dt = \lim_{\varepsilon \rightarrow 0} \int_\varepsilon^1 f(t) dt + \lim_{r \rightarrow \infty} \int_1^r f(t) dt,$$

provided that both limits exist.

9. The **non-negative real axis** is the positive real axis together with the origin.

10. **Theorem** Let p and q be polynomial functions such that

- the degree of q exceeds the degree of p by at least two
- any poles of p/q on the non-negative real axis are simple.

Then, for $0 < a < 1$,

$$\int_0^\infty \frac{p(t)}{q(t)} t^a dt = -(\pi e^{-\pi a i} \operatorname{cosec} \pi a) S - (\pi \cot \pi a) T,$$

where S is the sum of the residues of the function

$$f_1(z) = \frac{p(z)}{q(z)} \exp(a \operatorname{Log}_{2\pi}(z))$$

in $\mathbb{C}_{2\pi}$, and T is the sum of the residues of the function

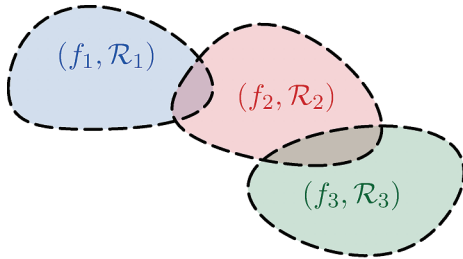
$$f_2(z) = \frac{p(z)}{q(z)} \exp(a \operatorname{Log} z)$$

on the positive real axis.

11. The notation (f, \mathcal{R}) denotes an analytic function f whose domain is the region \mathcal{R} .
12. The finite sequence of functions

$$(f_1, \mathcal{R}_1), (f_2, \mathcal{R}_2), \dots, (f_n, \mathcal{R}_n)$$

is called a **chain of functions** if $(f_{k+1}, \mathcal{R}_{k+1})$ is a direct analytic continuation of (f_k, \mathcal{R}_k) , for $k = 1, 2, \dots, n-1$.



Any two functions of a chain of functions are said to be **analytic continuations** of each other. If the two functions are not direct analytic continuations of each other, then they are said to be **indirect analytic continuations** of each other.

A chain of functions is **closed** if $\mathcal{R}_1 = \mathcal{R}_n$.

Unit C2 Zeros and extrema

Section 1 Winding numbers

1. Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a path lying in $\mathbb{C} - \{0\}$.

A **continuous argument function** for Γ is a continuous function

$$\theta : [a, b] \longrightarrow \mathbb{R}$$

such that, for each $t \in [a, b]$, $\theta(t)$ is an argument of $\gamma(t)$.

2. A continuous argument function θ for Γ satisfies

$$\frac{\gamma(t)}{|\gamma(t)|} = e^{i\theta(t)}, \quad \text{for } t \in [a, b].$$

3. **Theorem** Any path $\Gamma : \gamma(t)$ ($t \in [a, b]$) lying in $\mathbb{C} - \{0\}$ has a continuous argument function θ , which is unique apart from the addition of a constant term of the form $2\pi n$, where $n \in \mathbb{Z}$.

4. Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a path lying in $\mathbb{C} - \{0\}$. The **winding number** of Γ around 0 is

$$\text{Wnd}(\Gamma, 0) = \frac{1}{2\pi}(\theta(b) - \theta(a)),$$

where θ is any continuous argument function for Γ .

5. **Theorem** Let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a closed contour lying in $\mathbb{C} - \{0\}$. Then

$$\text{Wnd}(\Gamma, 0) = \frac{1}{2\pi i} \int_{\Gamma} \frac{1}{z} dz.$$

6. Let α be an arbitrary point in \mathbb{C} , and let $\Gamma : \gamma(t)$ ($t \in [a, b]$) be a path lying in $\mathbb{C} - \{\alpha\}$.

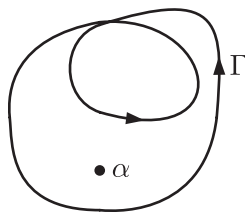
A **continuous argument function** for Γ relative to α is a continuous function $\theta_{\alpha} : [a, b] \longrightarrow \mathbb{R}$ such that, for each $t \in [a, b]$, $\theta_{\alpha}(t)$ is an argument of $\gamma(t) - \alpha$.

The **winding number** of Γ around α is

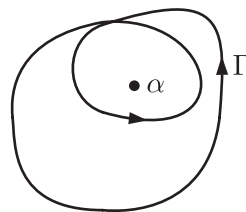
$$\text{Wnd}(\Gamma, \alpha) = \frac{1}{2\pi}(\theta_{\alpha}(b) - \theta_{\alpha}(a)),$$

where θ_{α} is any continuous argument function for Γ relative to α .

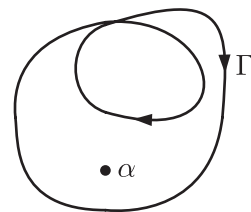
7. $\text{Wnd}(\Gamma, \alpha)$ can often be calculated from a sketch.



$$\text{Wnd}(\Gamma, \alpha) = 1$$



$$\text{Wnd}(\Gamma, \alpha) = 2$$



$$\text{Wnd}(\Gamma, \alpha) = -1$$

8. An equivalent definition of $\text{Wnd}(\Gamma, \alpha)$ is

$$\text{Wnd}(\Gamma, \alpha) = \text{Wnd}(\Gamma - \alpha, 0),$$

where

$$\Gamma - \alpha : \gamma(t) - \alpha \quad (t \in [a, b])$$

is the path Γ translated by $-\alpha$.

9. **Theorem** Let Γ be a closed path, and let D be an open disc in the complement of Γ . Then the function $\alpha \mapsto \text{Wnd}(\Gamma, \alpha)$ is constant on D .

Section 2 Locating zeros of analytic functions

1. **Theorem** Let f be an analytic function with a zero of order n at α . Then the function f'/f has a simple pole at α with

$$\text{Res}(f'/f, \alpha) = n.$$

2. The function f'/f is called the **logarithmic derivative** of f .
3. **Theorem** Let f be a function that is analytic on a simply connected region \mathcal{R} , and let Γ be a simple-closed contour in \mathcal{R} such that $f(z) \neq 0$, for $z \in \Gamma$. Then

$$\frac{1}{2\pi i} \int_{\Gamma} \frac{f'(z)}{f(z)} dz$$

is equal to the number of zeros of f inside Γ , counted according to their orders.

4. **Argument Principle** Let f be a function that is analytic on a simply connected region \mathcal{R} , and let Γ be a simple-closed contour in \mathcal{R} such that $f(z) \neq 0$, for $z \in \Gamma$. Then $\text{Wnd}(f(\Gamma), 0)$ is equal to the number of zeros of f inside Γ , counted according to their orders.
5. Let f be a function that is analytic on a simply connected region \mathcal{R} , and let Γ be a simple-closed contour in \mathcal{R} such that $f(z) \neq \beta$, for $z \in \Gamma$. Then $\text{Wnd}(f(\Gamma), \beta)$ is the number of zeros of the function $f - \beta$ inside Γ , counted according to their orders.
6. **Rouché's Theorem** Suppose that f and g are analytic functions on a simply connected region \mathcal{R} , and Γ is a simple-closed contour in \mathcal{R} with

$$|f(z) - g(z)| < |g(z)|, \quad \text{for } z \in \Gamma.$$

Then f has the same number of zeros as g inside Γ , each counted according to their orders.

7. The function g in Rouché's Theorem is referred to as a **dominant term** for f on Γ .
8. Suppose that f is a function that satisfies $\overline{f(z)} = f(\bar{z})$, for all points z in its domain. Then $f(z) = 0$ if and only if $f(\bar{z}) = 0$, so non-real zeros of f occur in complex conjugate pairs.

Section 3 Local behaviour of analytic functions

1. **Open Mapping Theorem** Let f be a function that is analytic and non-constant on a region \mathcal{R} , and let G be an open subset of \mathcal{R} . Then $f(G)$ is open.
2. Let f be a function that is analytic and non-constant on a region \mathcal{R} . Then $f(\mathcal{R})$ is also a region.
3. Let f be a function that is analytic on a region \mathcal{R} , and let $\alpha \in \mathcal{R}$. Then f is **n -to-one near α** if there is a region \mathcal{S} inside \mathcal{R} with $\alpha \in \mathcal{S}$ such that for each point w in $f(\mathcal{S}) - \{f(\alpha)\}$ there are exactly n points z in $\mathcal{S} - \{\alpha\}$ that satisfy $f(z) = w$.
4. **Local Mapping Theorem** Let f be a function that is analytic on a region \mathcal{R} , and let $\alpha \in \mathcal{R}$. Suppose that the Taylor series about α for f has the form

$$f(z) = f(\alpha) + a_n(z - \alpha)^n + a_{n+1}(z - \alpha)^{n+1} + \cdots,$$

where $n \geq 1$ and $a_n \neq 0$. Then f is n -to-one near α .

5. Let f be a function that is analytic on a region \mathcal{R} , and let $\alpha \in \mathcal{R}$. Suppose that

$$f'(\alpha) = f''(\alpha) = \cdots = f^{(n-1)}(\alpha) = 0, \text{ but } f^{(n)}(\alpha) \neq 0,$$

where $n \geq 1$. Then f is n -to-one near α .

6. **Inverse Function Rule** Let f be a one-to-one analytic function whose domain is a region \mathcal{R} . Then f^{-1} is analytic on $f(\mathcal{R})$ and

$$(f^{-1})'(\beta) = \frac{1}{f'(f^{-1}(\beta))}, \quad \text{for } \beta \in f(\mathcal{R}).$$

7. The restrictions of the functions \tan and \sin to the region $\{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$ have analytic inverse functions \tan^{-1} and \sin^{-1} with derivatives

$$(\tan^{-1})'(z) = \frac{1}{1+z^2} \quad \text{and} \quad (\sin^{-1})'(z) = \frac{1}{\sqrt{1-z^2}}.$$

8. **Strategy for inverting a Taylor series** Given the Taylor series about α for f ,

$$f(z) = \sum_{n=0}^{\infty} a_n(z - \alpha)^n,$$

where $a_1 = f'(\alpha) \neq 0$, we can find the Taylor series about $\beta = f(\alpha)$ for f^{-1} ,

$$f^{-1}(w) = \sum_{n=0}^{\infty} b_n(w - \beta)^n,$$

by putting $b_0 = \alpha$ and equating the powers of $(z - \alpha)$ in the identity

$$\begin{aligned} z - \alpha &= b_1(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots) \\ &\quad + b_2(a_1(z - \alpha) + a_2(z - \alpha)^2 + \cdots)^2 + \cdots \end{aligned}$$

to obtain equations for b_1, b_2, \dots in terms of a_1, a_2, \dots .

Section 4 Extreme values of analytic functions

1. Let f be a function that is defined on a region \mathcal{R} . Then the function $|f|$ has a **local maximum** at a point $\alpha \in \mathcal{R}$ if there is some $r > 0$ such that $\{z : |z - \alpha| < r\} \subseteq \mathcal{R}$ and

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } |z - \alpha| < r.$$

2. **Local Maximum Principle** Let f be a function that is analytic and non-constant on a region \mathcal{R} . Then the function $|f|$ has no local maxima on \mathcal{R} .

3. The **closure** \overline{A} of a set A in \mathbb{C} is

$$\overline{A} = \text{int } A \cup \partial A.$$

4. **Maximum Principle** Let f be a function that is analytic and non-constant on a bounded region \mathcal{R} , and continuous on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \leq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

with strict inequality for any $z \in \mathcal{R}$.

5. **Minimum Principle** Let f be a function that is analytic and non-constant on a bounded region \mathcal{R} , and continuous and non-zero on $\overline{\mathcal{R}}$. Then there exists $\alpha \in \partial\mathcal{R}$ such that

$$|f(z)| \geq |f(\alpha)|, \quad \text{for } z \in \overline{\mathcal{R}},$$

with strict inequality for any $z \in \mathcal{R}$.

6. **Boundary Uniqueness Theorem** Let f and g be functions that are analytic on a bounded region \mathcal{R} and continuous on $\overline{\mathcal{R}}$. If $f = g$ on $\partial\mathcal{R}$, then $f = g$ on \mathcal{R} .

7. **Schwarz's Lemma** Let f be a function that is analytic on the open disc $\{z : |z| < R\}$, with $f(0) = 0$, and suppose that

$$|f(z)| \leq M, \quad \text{for } |z| < R.$$

Then

$$|f(z)| \leq (M/R)|z|, \quad \text{for } |z| < R.$$

Section 5 Uniform convergence

1. A sequence of functions (f_n) **converges pointwise** (to a **limit function** f) on a set E if, for each $z \in E$,

$$\lim_{n \rightarrow \infty} f_n(z) = f(z).$$

2. A sequence of functions (f_n) **converges uniformly** (to a **limit function** f) on a set E if, for each $\varepsilon > 0$, there is an integer N such that

$$|f_n(z) - f(z)| < \varepsilon, \quad \text{for all } n > N \text{ and all } z \in E.$$

We also say that (f_n) is **uniformly convergent** on E , with limit function f .

3. In the definition of uniform convergence the choice of N depends only on ε – the same N works for all $z \in E$. By contrast, for pointwise convergence N depends on ε and on z .
4. If (f_n) converges uniformly to f on E , then it converges uniformly to f on any subset of E .

If (f_n) converges uniformly to f on E , then (f_n) converges pointwise to f on E .

5. **Strategy for proving uniform convergence** To prove that a sequence of functions (f_n) is uniformly convergent on a set E , proceed as follows.

- (1) Determine the limit function f by evaluating

$$f(z) = \lim_{n \rightarrow \infty} f_n(z), \quad \text{for } z \in E.$$

- (2) Find a null sequence (a_n) of positive terms such that

$$|f_n(z) - f(z)| \leq a_n, \quad \text{for } n = 1, 2, \dots \text{ and all } z \in E.$$

6. If (ϕ_n) is a sequence of functions, then the series of functions

$$\sum_{n=1}^{\infty} \phi_n = \phi_1 + \phi_2 + \dots$$

converges pointwise on a set E if the sequence of **partial sum functions** (f_n) , where

$$f_n(z) = \phi_1(z) + \phi_2(z) + \dots + \phi_n(z), \quad n = 1, 2, \dots,$$

converges pointwise on E .

The series of functions **converges uniformly** on a set E , or is **uniformly convergent** on E , if the sequence of partial sum functions converges uniformly on E .

The limit function f of the sequence (f_n) is called the **sum function**

of $\sum_{n=1}^{\infty} \phi_n$ on E , written

$$f(z) = \sum_{n=1}^{\infty} \phi_n(z) \quad (z \in E).$$

7. **Weierstrass' M-test** Let (ϕ_n) be a sequence of functions defined on a set E , and suppose that there is a sequence of positive numbers (M_n) such that

1. $|\phi_n(z)| \leq M_n$, for $n = 1, 2, \dots$ and all $z \in E$

2. $\sum_{n=1}^{\infty} M_n$ is convergent.

Then the series $\sum_{n=1}^{\infty} \phi_n$ is uniformly convergent on E .

8. Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series with disc of convergence $\{z : |z| < R\}$,

where $R > 0$. Then the power series is uniformly convergent on each closed disc $\{z : |z| \leq r\}$, where $0 < r < R$.

9. **Weierstrass' Theorem** Let (f_n) be a sequence of functions, each of which is analytic on a region \mathcal{R} , and suppose that (f_n) converges uniformly to a function f on each closed disc in \mathcal{R} . Then

- (a) f is analytic on \mathcal{R}
- (b) the sequence (f'_n) converges uniformly to f' on each closed disc in \mathcal{R} .

10. The **zeta function** is the function

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \frac{1}{4^z} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^z} \quad (\operatorname{Re} z > 1).$$

11. Some values of the zeta function can be calculated using residue calculus, such as

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \quad \text{and} \quad \zeta(4) = \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Most values of the function ζ can be found only approximately.

12. The zeta function ζ is analytic on $\{z : \operatorname{Re} z > 1\}$ and

$$\zeta'(z) = - \sum_{n=2}^{\infty} \frac{\log n}{n^z} \quad (\operatorname{Re} z > 1).$$

Section 6 Special functions

1. The **gamma function** is the function

$$\Gamma(z) = \int_0^{\infty} e^{-t} t^{z-1} dt \quad (\operatorname{Re} z > 1).$$

2. **Theorem** The gamma function is an analytic function and

$$\Gamma'(z) = \int_0^{\infty} e^{-t} t^{z-1} \log t dt \quad (\operatorname{Re} z > 1).$$

3. **Theorem** The gamma function has an analytic continuation Γ to $\mathbb{C} - \{0, -1, -2, \dots\}$ with simple poles at $0, -1, -2, \dots$ such that

$$\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}, \quad \text{for } k = 0, 1, 2, \dots$$

4. **Functional equation for the gamma function**

$$\Gamma(z+1) = z\Gamma(z), \quad \text{for } z \in \mathbb{C} - \{0, -1, -2, \dots\}.$$

5. $\Gamma(n+1) = n!$, for $n = 1, 2, \dots$

6. **Gaussian integral**

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}.$$

7. **Theorem** $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Unit C3 Conformal mappings

Section 1 Linear and reciprocal functions

1. A function of the form $f(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, is called a **linear function**.
2. A **scaling** is a function of the form $f(z) = rz$, where $r > 0$.
A **rotation** about 0 (through the angle $\theta \in \mathbb{R}$) is a function of the form $f(z) = e^{i\theta}z$.
A **translation** is a function of the form $f(z) = z + b$, where $b \in \mathbb{C}$.
3. **Theorem** Linear functions map lines onto lines and circles onto circles. Furthermore:
 - (a) given any two lines L_1 and L_2 , there is a linear function that maps L_1 onto L_2
 - (b) given any two circles C_1 and C_2 , there is a linear function that maps C_1 onto C_2 .
4. The **reciprocal function** is the function

$$f(z) = \frac{1}{z} \quad (z \in \mathbb{C} - \{0\}).$$

5. **Strategy for finding an equation for the image of a path under the reciprocal function** To find an equation for the image $f(\Gamma)$ of a path Γ under $f(z) = 1/z$, apply the following steps.
 - (1) Write down an equation that relates the x - and y -coordinates of all points $x + iy$ on Γ .
 - (2) Replace x by $\frac{u}{u^2 + v^2}$ and y by $\frac{-v}{u^2 + v^2}$.
 - (3) Simplify the resulting equation to obtain an equation that relates the u - and v -coordinates of all points $u + iv$ on the image $f(\Gamma)$.
6. **Theorem** Every line or circle has an equation of the form

$$a(x^2 + y^2) + bx + cy + d = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2 + c^2 > 4ad$.

Conversely, any such equation represents a line or circle. Also:

- (a) the equation represents a line if and only if $a = 0$
 - (b) the line or circle passes through the origin if and only if $d = 0$.
7. **Theorem** The reciprocal function maps the set of non-zero points on the line or circle

$$a(x^2 + y^2) + bx + cy + d = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2 + c^2 > 4ad$, onto the set of non-zero points on the line or circle

$$d(u^2 + v^2) + bu - cv + a = 0,$$

where $a, b, c, d \in \mathbb{R}$ and $b^2 + (-c)^2 > 4da$.

8. The **extended complex plane** $\widehat{\mathbb{C}}$ is the union of the ordinary complex plane \mathbb{C} and one extra element, which is called the **point at infinity**, denoted by ∞ . Thus $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$.
9. Given a function f with a pole at α , we can extend the definition of f to α by defining $f(\alpha) = \infty$.
10. Given a rational function f , and a point $\beta \in \widehat{\mathbb{C}}$, we write

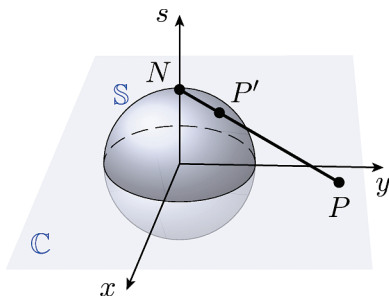
$$f(z) \rightarrow \beta \text{ as } z \rightarrow \infty$$

to mean that

$$f(1/w) \rightarrow \beta \text{ as } w \rightarrow 0.$$

If this holds, then we can extend the domain of f to include the point ∞ by defining $f(\infty) = \beta$.

11. Let L be a line. Then the set $L \cup \{\infty\}$ is called an **extended line**.
12. A **generalised circle** is a circle or an extended line.
13. **Theorem** Linear functions and the reciprocal function have the following properties:
 - (a) they are one-to-one mappings from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$
 - (b) they map generalised circles onto generalised circles.
14. The sphere \mathbb{S} in three-dimensional space centred at the origin of radius 1 is called the **Riemann sphere**. The point $N = (0, 0, 1)$ is called the *North Pole*.
15. Consider the complex plane \mathbb{C} embedded in three-dimensional space in such a way that each complex number $x + iy$ is represented by the point $(x, y, 0)$ in the (x, y) -plane. Each line that joins a point P in the complex plane to the North Pole intersects the Riemann sphere at a point P' , say, and vice versa.



The function $\pi: \mathbb{S} \rightarrow \widehat{\mathbb{C}}$ that projects the point P' on the Riemann sphere to the associated point P in the complex plane, and maps N to ∞ , is called **stereographic projection**.

16. **Theorem**
 - (a) Stereographic projection maps circles on the Riemann sphere \mathbb{S} onto generalised circles in $\widehat{\mathbb{C}}$, and every generalised circle in $\widehat{\mathbb{C}}$ is the image of some circle on \mathbb{S} .
 - (b) Stereographic projection preserves angles.

Section 2 Möbius transformations

1. A function of the form

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

is called a **Möbius transformation**.

2. **Theorem** Every Möbius transformation is analytic and conformal.

3. **Convention** Each Möbius transformation is considered to be extended (see items 9 and 10 in Section 1) to give a function from $\widehat{\mathbb{C}}$ to $\widehat{\mathbb{C}}$.

4. Each Möbius transformation is either a linear function, or a composition of linear functions and the reciprocal function.

5. **Theorem**

- (a) Möbius transformations are one-to-one mappings from $\widehat{\mathbb{C}}$ onto $\widehat{\mathbb{C}}$.
- (b) Möbius transformations map generalised circles onto generalised circles.

6. **Inverse function of a Möbius transformation** The Möbius transformation

$$f(z) = \frac{az + b}{cz + d}, \quad \text{where } a, b, c, d \in \mathbb{C} \text{ and } ad - bc \neq 0,$$

has inverse function

$$f^{-1}(w) = \frac{dw - b}{-cw + a}.$$

The inverse function f^{-1} is itself a Möbius transformation.

7. **Group properties** The set of Möbius transformations has the following properties.

Closure If f and g are Möbius transformations, then so is $f \circ g$.

Identity The identity function on $\widehat{\mathbb{C}}$ is a Möbius transformation.

Inverses Each Möbius transformation f has an inverse function f^{-1} that is also a Möbius transformation.

Associativity If f , g and h are Möbius transformations, then

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

8. A **fixed point** of a Möbius transformation f is a point $\alpha \in \widehat{\mathbb{C}}$ for which $f(\alpha) = \alpha$.

9. **Theorem** Each Möbius transformation, other than the identity function, has either one or two fixed points in $\widehat{\mathbb{C}}$.

10. **Theorem** If two Möbius transformations f and g satisfy $f(z) = g(z)$ for three or more points z in $\widehat{\mathbb{C}}$, then $f = g$.

11. **Theorem** Given two triples of three distinct points α, β, γ and α', β', γ' in $\widehat{\mathbb{C}}$, there is a unique Möbius transformation that maps

$$\alpha \text{ to } \alpha', \quad \beta \text{ to } \beta' \quad \text{and} \quad \gamma \text{ to } \gamma'.$$

12. **Implicit Formula for Möbius Transformations** Given two triples of three distinct points α, β, γ and α', β', γ' in $\widehat{\mathbb{C}}$, the unique Möbius transformation that sends α to α' , β to β' and γ to γ' is the function f that maps z to w , where

$$\frac{(z - \alpha)(\beta - \gamma)}{(z - \gamma)(\beta - \alpha)} = \frac{(w - \alpha')(\beta' - \gamma')}{(w - \gamma')(\beta' - \alpha')}.$$

13. **Explicit Formula for Möbius Transformations** Given a triple of three distinct points α, β, γ in $\widehat{\mathbb{C}}$, the unique Möbius transformation that sends α to 0, β to 1 and γ to ∞ is

$$f(z) = \frac{(z - \alpha)(\beta - \gamma)}{(z - \gamma)(\beta - \alpha)}.$$

14. **Theorem** Given any two generalised circles C_1 and C_2 , there is a Möbius transformation that maps C_1 onto C_2 .

Section 3 Images of generalised circles

- Every generalised circle is completely determined by the positions of any three of its points.
- Three-point trick** The image of a generalised circle under a Möbius transformation can be determined by finding the images of three points on the generalised circle.
- Substitution method** The image of a generalised circle C under a Möbius transformation f can be determined by substituting $z = f^{-1}(w)$ into the equation for C .
- Any generalised circle C can be represented by an equation of the form

$$|z - \alpha| = k|z - \beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0,$$

called the **Apollonian form** of an equation for C . If $k = 1$, then the equation represents a line which, by convention, includes ∞ (so it represents an extended line).

- Let C be a generalised circle. Then α and β are **inverse points with respect to C** if
 - either* α and β are equal and lie on C
 - or* there exists a Möbius transformation f that maps α to 0, β to ∞ , and C onto the unit circle.
- Theorem** The points α and β in $\widehat{\mathbb{C}}$ are distinct inverse points with respect to a generalised circle C if and only if
 - either* both α and β belong to \mathbb{C} , and C has the equation (in Apollonian form)

$$|z - \alpha| = k|z - \beta|, \quad \text{for some } k > 0$$

- or* one of the points (β say) is ∞ , and C has the equation

$$|z - \alpha| = r, \quad \text{for some } r > 0.$$

7. The points α and β in \mathbb{C} are inverse points with respect to an extended line L if and only if α is the reflection of β in L .
8. The centre α of a circle C and the point ∞ are inverse points with respect to C .
9. **Theorem** Let f be a Möbius transformation. If α and β are inverse points with respect to a generalised circle C , then $f(\alpha)$ and $f(\beta)$ are inverse points with respect to $f(C)$.
10. **Inverse points method** The image of a generalised circle C under a Möbius transformation can be determined by finding the images of a pair of inverse points with respect to C .
11. **Theorem** Let C be the generalised circle with equation
$$|z - \alpha| = k|z - \beta|, \quad \text{where } \alpha, \beta \in \mathbb{C} \text{ and } k > 0.$$
 - (a) If $k \neq 1$, then C is the circle centred at λ of radius r , where
$$\lambda = \frac{\alpha - k^2\beta}{1 - k^2} \quad \text{and} \quad r = \frac{k|\alpha - \beta|}{|1 - k^2|}.$$
Also, λ lies on the line through α and β , and
$$(\alpha - \lambda)\overline{(\beta - \lambda)} = r^2.$$
 - (b) If $k = 1$, then C is the extended line through $\frac{1}{2}(\alpha + \beta)$ that is perpendicular to the line through α and β .
12. **Existence of inverse points** Let C be a generalised circle, and let β be an arbitrary point of $\widehat{\mathbb{C}}$. Then there is a unique point α such that α and β are inverse points with respect to C .

Section 4 Transforming regions

1. An **open disc centred at ∞** is a set of the form

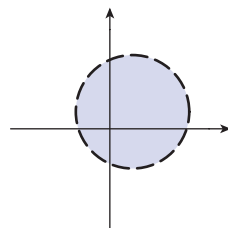
$$\{z : |z| > M\} \cup \{\infty\},$$

where $M > 0$.

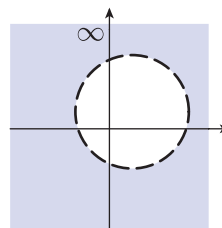
2. Let A be a subset of $\widehat{\mathbb{C}}$, and let $\alpha \in \widehat{\mathbb{C}}$. Then α is a **boundary point** in $\widehat{\mathbb{C}}$ of A if each open disc centred at α contains at least one point of A and at least one point of $\widehat{\mathbb{C}} - A$.

The set of boundary points in $\widehat{\mathbb{C}}$ of A forms the **boundary** in $\widehat{\mathbb{C}}$ of A .

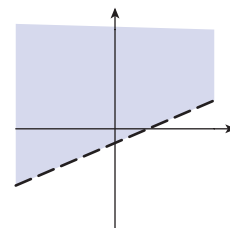
3. Each generalised circle separates $\widehat{\mathbb{C}}$ into two parts, which together form the complement in $\widehat{\mathbb{C}}$ of the generalised circle. Each of these parts is called a **generalised open disc**. There are three types, as follows.



open disc

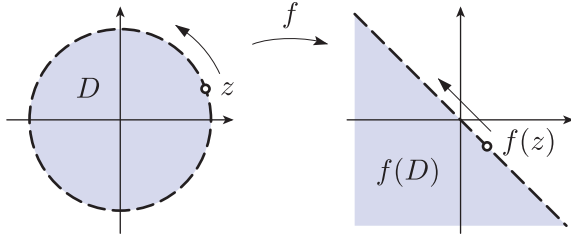


outside of a circle
together with ∞



open half-plane

4. **Theorem** Let f be a Möbius transformation, and let D be a generalised open disc with boundary C in $\widehat{\mathbb{C}}$. Then $f(D)$ is a generalised open disc with boundary $f(C)$ in $\widehat{\mathbb{C}}$.
5. Let f be a Möbius transformation, and let D and $f(D)$ be generalised open discs (or they are both *lunes*; see item 7). Then, as a point z traverses the boundary of D with D on its left, the image point $f(z)$ traverses the boundary of $f(D)$ with $f(D)$ on its left. (A similar statement holds with ‘right’ in place of ‘left’ for *both* points z and $f(z)$.)

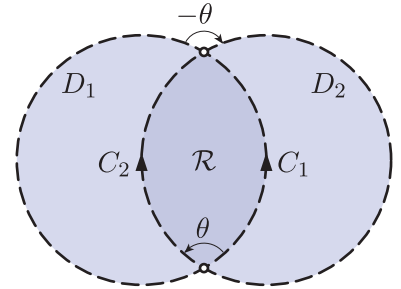


6. Any one-to-one analytic mapping from a region \mathcal{R} onto a region \mathcal{S} is a one-to-one conformal mapping from \mathcal{R} onto \mathcal{S} .
7. A **lune** is a set in $\widehat{\mathbb{C}}$ formed from the intersection of two generalised open discs whose boundaries in $\widehat{\mathbb{C}}$, which are generalised circles, intersect at exactly two points.

The two intersection points are called the **vertices** of the lune.

8. Let D_1 and D_2 be generalised open discs such that $\mathcal{R} = D_1 \cap D_2$ is a lune. Let C_1 and C_2 be smooth paths that traverse the boundaries in \mathbb{C} of D_1 and D_2 , respectively. We choose the directions of C_1 and C_2 such that D_1 lies to the *left* of C_1 , and D_2 lies to the *right* of C_2 .

The **angle** of the lune \mathcal{R} is the absolute value of the angle from C_1 to C_2 at a vertex (not ∞) of \mathcal{R} .



9. **Strategy for mapping lunes** Let \mathcal{R} be a lune with vertices α and β , and let Γ be a smooth path in the complex plane that traverses one of the boundary arcs of \mathcal{R} from α to β , excluding the endpoints, such that \mathcal{R} lies to the *left* of Γ . Let \mathcal{R}' , α' , β' and Γ' be defined in a similar way.

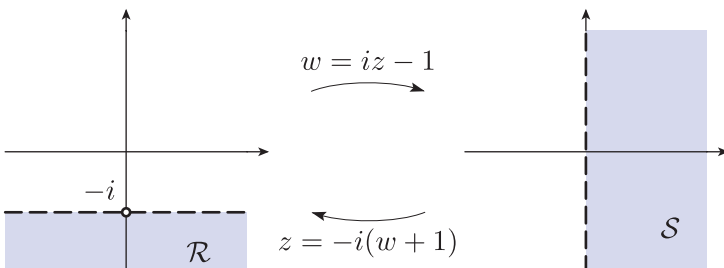
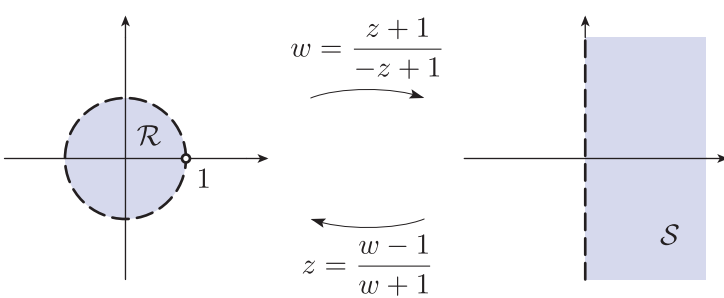
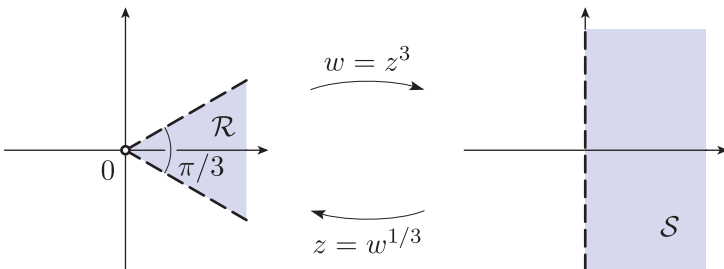
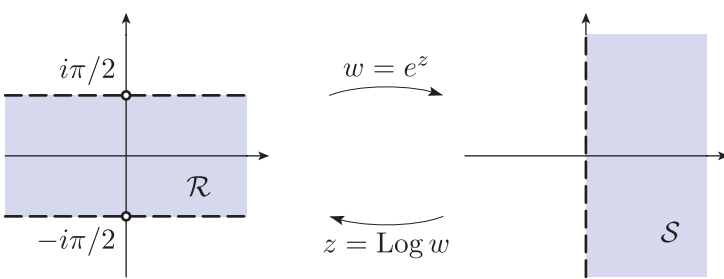
(Alternatively, we can replace ‘left’ by ‘right’ for *both* paths Γ and Γ' .)

Suppose that the angles of \mathcal{R} and \mathcal{R}' are equal.

To find a Möbius transformation f that maps \mathcal{R} onto \mathcal{R}' , carry out the following steps.

- (1) Define $f(\alpha) = \alpha'$ and $f(\beta) = \beta'$.
- (2) Choose any points γ on Γ and γ' on Γ' , and define $f(\gamma) = \gamma'$.
- (3) Use the Implicit or Explicit Formula for Möbius Transformations to determine f .

10. Standard conformal mappings

Basic region	Mapping	Example
Open half-plane	Linear function	
Open disc	Möbius transformation	
Open sector with vertex at the origin	Power function	
Open horizontal strip	Exponential function	

11. **Theorem** The function \tan is a one-to-one conformal mapping from

$$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$$

onto

$$\mathcal{S} = \mathbb{C} - \{iv : v \in \mathbb{R}, |v| \geq 1\}$$

with inverse function

$$\tan^{-1} w = \frac{1}{2i} \operatorname{Log} \left(\frac{1+iw}{1-iw} \right).$$

12. **Theorem** The function \sin is a one-to-one conformal mapping from

$$\mathcal{R} = \{z : -\pi/2 < \operatorname{Re} z < \pi/2\}$$

onto

$$\mathcal{S} = \mathbb{C} - \{u \in \mathbb{R} : |u| \geq 1\}$$

with inverse function

$$\sin^{-1} w = \frac{1}{i} \operatorname{Log} \left(iw + \sqrt{1-w^2} \right).$$

Unit D1 Fluid flows

Section 1 Setting up the model

1. Basic fluid flow model We assume that

- the flow is two-dimensional
- the fluid forms a continuum, and any variation of the flow velocity within this continuum is continuous
- the flow is steady.

With these assumptions, we can represent the flow velocity at all times by a continuous complex function q , called the **velocity function**, whose domain is the region occupied by the fluid.

2. At a point z where the fluid is at rest (has zero speed), the velocity function satisfies $q(z) = 0$. Such a point is called a **stagnation point** of the flow.
3. If q is a constant function, then the associated flow is a **uniform flow** or **uniform stream**.
4. A **streamline** (or **flow line**) through the point z_0 , for a flow with velocity function q , is a smooth path $\Gamma : \gamma(t)$ ($t \in I$) such that
 - $\gamma'(t) = q(\gamma(t))$, for $t \in I$
 - $z_0 = \gamma(t_0)$, for some $t_0 \in I$.

If $q(z_0) = 0$ (that is, if z_0 is a stagnation point), then the point z_0 is a **degenerate streamline**, with constant parametrisation

$$\gamma(t) = z_0 \quad (t \in I).$$

5. The component of $q(z)$ in the direction specified by $e^{i\theta}$ is

$$q_\theta(z) = \operatorname{Re}(\overline{q(z)}e^{i\theta}).$$

The component of $q(z)$ in the direction specified by $e^{i(\theta-\pi/2)}$ is

$$q_{(\theta-\pi/2)}(z) = \operatorname{Im}(\overline{q(z)}e^{i\theta}).$$

6. The **conjugate velocity function** \bar{q} has the same domain as q and has rule

$$\bar{q}(z) = \overline{q(z)}.$$

7. Let $\Gamma : \gamma(t)$ ($t \in I$) be a smooth path. Then γ is a **unit-speed parametrisation** if

$$|\gamma'(t)| = 1, \quad \text{for } t \in I.$$

8. **Theorem** Let $\Gamma : \gamma_1(t)$ ($t \in [a, b]$) be a smooth path of length L . Then there is another parametrisation $\gamma(s)$ ($s \in [0, L]$) of Γ such that

$$|\gamma'(s)| = 1, \quad \text{for } 0 \leq s \leq L.$$

9. Let $\Gamma : \gamma(s)$ ($s \in [0, L]$) be a smooth path with unit-speed parametrisation which lies in the domain of a flow with velocity function q . Then, for each $s \in [0, L]$, the velocity $q(\gamma(s))$ has

- **tangential component** $q_T(s)$ in the direction specified by $\gamma'(s)$
- **normal component** $q_N(s)$ in the direction specified by $-i\gamma'(s)$.

10. Let $\Gamma : \gamma(s)$ ($s \in [0, L]$) be a smooth path with unit-speed parametrisation which lies in the domain of a flow with continuous velocity function q .

- The **circulation** of q along Γ is

$$C_\Gamma = \int_0^L q_T(s) ds.$$

- The **flux** of q across Γ is

$$\mathcal{F}_\Gamma = \int_0^L q_N(s) ds.$$

11. **Circulation and Flux Contour Integral** For any contour Γ in the domain of a flow with continuous velocity function q , we have

$$C_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \bar{q}(z) dz.$$

12. A flow with continuous velocity function q and domain a region \mathcal{R} is

- **locally circulation-free** if $C_\Gamma = 0$ for each simple-closed contour Γ in \mathcal{R} whose inside also lies in \mathcal{R}
- **locally flux-free** if $\mathcal{F}_\Gamma = 0$ for each simple-closed contour Γ in \mathcal{R} whose inside also lies in \mathcal{R} .

13. An **ideal flow** is a fluid flow, defined by a continuous velocity function on a region, that is locally circulation-free and locally flux-free.

14. A steady two-dimensional fluid flow with continuous velocity function q on a region \mathcal{R} is an ideal flow if and only if

$$\int_\Gamma \bar{q}(z) dz = 0,$$

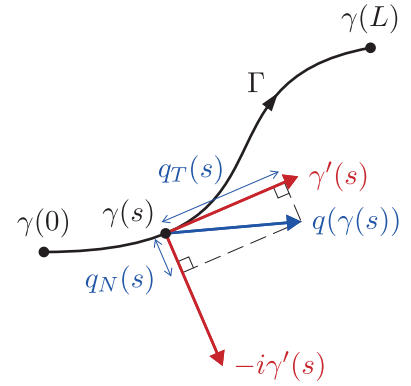
for each simple-closed contour Γ in \mathcal{R} whose inside also lies in \mathcal{R} .

15. **Theorem** A steady two-dimensional fluid flow with continuous velocity function q on a region \mathcal{R} is an ideal flow if and only if its conjugate velocity function \bar{q} is analytic on \mathcal{R} .

16. Let q be a velocity function for an ideal flow with flow region \mathcal{R} , and let D be a punctured open disc in \mathcal{R} with centre α . Then

- α is a **source** of strength \mathcal{F} if $\mathcal{F}_\Gamma = \mathcal{F} > 0$ for each simple-closed contour Γ in D that surrounds α
- α is a **sink** of strength $|\mathcal{F}|$ if $\mathcal{F}_\Gamma = \mathcal{F} < 0$ for each simple-closed contour Γ in D that surrounds α
- α is a **vortex** of strength $|\mathcal{C}|$ if $C_\Gamma = \mathcal{C} \neq 0$ for each simple-closed contour Γ in D that surrounds α .

An **anticlockwise vortex** is a vortex with $\mathcal{C} > 0$, and a **clockwise vortex** is a vortex with $\mathcal{C} < 0$.



17. **Theorem** Let q be a continuous velocity function on a region \mathcal{R} , and suppose that $q_1 = \operatorname{Re} q$ and $q_2 = \operatorname{Im} q$ have partial derivatives with respect to x and y that are continuous on \mathcal{R} .

The flow with velocity function q is

- (a) locally circulation-free if and only if

$$\frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial y} = 0 \quad \text{on } \mathcal{R}$$

- (b) locally flux-free if and only if

$$\frac{\partial q_1}{\partial x} + \frac{\partial q_2}{\partial y} = 0 \quad \text{on } \mathcal{R}.$$

Section 2 Complex potential functions

1. Let q be the velocity function for an ideal flow with domain a region \mathcal{R} . A function Ω that is a primitive of \bar{q} on a region $\mathcal{S} \subseteq \mathcal{R}$ (that is, $\Omega'(z) = \bar{q}(z)$, for $z \in \mathcal{S}$) is called a **complex potential function** for the flow.

Such a complex potential function Ω always exists on any simply connected region contained in \mathcal{R} , by the Primitive Theorem.

2. If Ω is a complex potential function for an ideal flow, and Ω is defined on a region \mathcal{S} , then

$$\mathcal{C}_\Gamma + i\mathcal{F}_\Gamma = \int_\Gamma \Omega'(z) dz = \Omega(\beta) - \Omega(\alpha),$$

where Γ is any contour lying in \mathcal{S} with initial point α and final point β . Then

$$\mathcal{C}_\Gamma = \operatorname{Re} \Omega(\beta) - \operatorname{Re} \Omega(\alpha) \quad \text{and} \quad \mathcal{F}_\Gamma = \operatorname{Im} \Omega(\beta) - \operatorname{Im} \Omega(\alpha).$$

3. **Theorem** Suppose that an ideal flow is defined on a region \mathcal{R} , and Ω is a complex potential function for this flow on a region $\mathcal{S} \subseteq \mathcal{R}$.

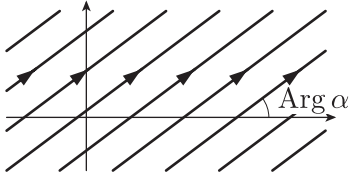
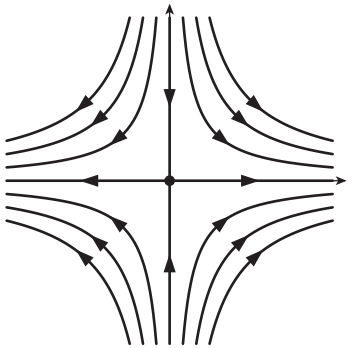
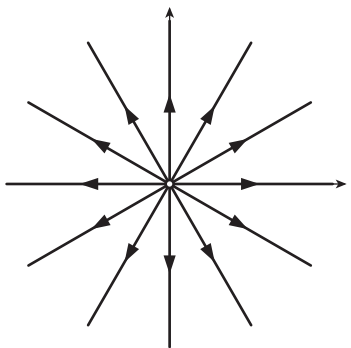
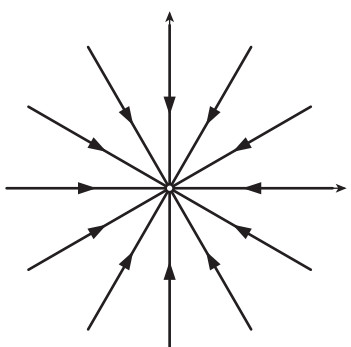
Then the streamlines for the flow within \mathcal{S} are the smooth paths with equations of the form $\operatorname{Im} \Omega(z) = k$, for some real constant k .

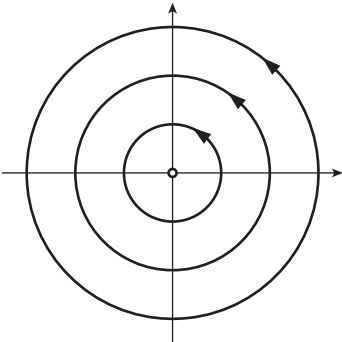
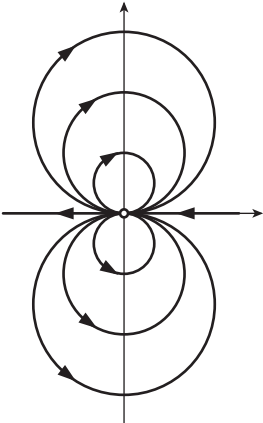
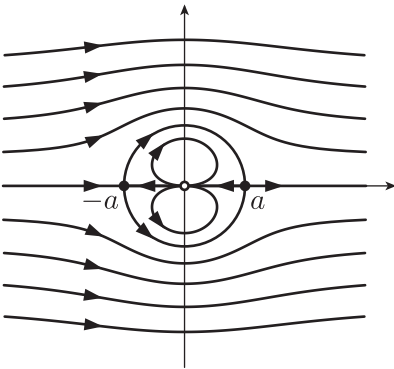
4. Each point of a flow region has just one streamline through it.
5. A stagnation point is a degenerate streamline consisting of a single point, so no *other* streamline passes through a stagnation point.
6. Let $\Omega(z) = \Phi(z) + i\Psi(z)$ be a complex potential function for an ideal flow. Then the function $\Psi = \operatorname{Im} \Omega$ is called a **stream function** for the flow. The streamlines for the flow within the domain of Ψ are the smooth paths with equations of the form

$$\Psi(z) = k,$$

for some real constant k .

6. Examples of fluid flows

Flow type	Velocity function	Streamlines
Uniform flow	$q(z) = \alpha,$ $\alpha \neq 0$	
Flow near a stagnation point	$q(z) = \bar{z}$	
Flow with a source	$q(z) = \frac{1}{\bar{z}} = \frac{z}{ z ^2}$	
Flow with a sink	$q(z) = -\frac{1}{\bar{z}} = -\frac{z}{ z ^2}$	

Flow type	Velocity function	Streamlines
Flow around a vortex	$q(z) = \frac{i}{z} = \frac{iz}{ z ^2}$	
Flow due to a doublet	$q(z) = -\frac{1}{\bar{z}^2} = -\frac{z^2}{ z ^4}$	
Flow due to a doublet in a uniform stream	$q(z) = 1 - \frac{a^2}{\bar{z}^2},$ $a > 0$	

7. In any fluid flow there is no flux across a streamline, so we can treat any streamline as the solid boundary of a flow lying on either side of that streamline.

Section 3 The Joukowski functions

1. The function

$$J(z) = z + \frac{1}{z} \quad (z \in \mathbb{C} - \{0\})$$

is called the basic **Joukowski function**.

2. **Theorem** The function $J(z) = z + 1/z$ has the following properties.

- (a) J maps the circle $\{z : |z| = 1\}$ onto the line segment $[-2, 2]$, with $J(1) = 2$ and $J(-1) = -2$.
- (b) J maps the region $\{z : |z| > 1\}$ conformally onto the region $\mathbb{C} - [-2, 2]$.
- (c) The restriction of J to $\{z : |z| > 1\}$ has inverse function

$$J^{-1}(w) = \frac{1}{2}(w + w\sqrt{1 - 4/w^2}) \quad (w \in \mathbb{C} - [-2, 2]).$$
- (d) J has non-vanishing derivative at all points of $\mathbb{C} - \{0\}$ except $z = \pm 1$.

3. We extend the family of Joukowski functions to include

$$J_\alpha(z) = z + \frac{\alpha^2}{z} \quad (z \in \mathbb{C} - \{0\}),$$

where α is any non-zero complex number. In particular, $J_1 = J$.

4. $L(-2\alpha, 2\alpha)$ denotes the line segment from -2α to 2α .

5. **Theorem** For $\alpha \in \mathbb{C} - \{0\}$, the function J_α has the following properties.

- (a) J_α maps the circle $\{z : |z| = |\alpha|\}$ onto the line segment $L(-2\alpha, 2\alpha)$, with $J(\alpha) = 2\alpha$ and $J(-\alpha) = -2\alpha$.
- (b) J_α maps the region $\{z : |z| > |\alpha|\}$ conformally onto the region $\mathbb{C} - L(-2\alpha, 2\alpha)$.
- (c) The restriction of J_α to $\{z : |z| > |\alpha|\}$ has inverse function

$$J_\alpha^{-1}(w) = \frac{1}{2}(w + w\sqrt{1 - 4\alpha^2/w^2}) \quad (w \in \mathbb{C} - L(-2\alpha, 2\alpha)).$$
- (d) J_α has non-vanishing derivative at all points of $\mathbb{C} - \{0\}$ except $z = \pm\alpha$.

6. Let R_ϕ denote the rotation about 0 through the angle $\phi = -\text{Arg } \alpha$. Then

$$J_\alpha = R_\phi^{-1} \circ J_{|\alpha|} \circ R_\phi.$$

Section 4 Flow past an obstacle

1. Let $a > 0$. Then K_a and C_a denote the sets

$$K_a = \{z : |z| \leq a\},$$

$$C_a = \partial K_a = \{z : |z| = a\}.$$

2. **Theorem** For $a > 0$, $c \in \mathbb{R}$, the ideal flow with velocity function

$$q_{a,c}(z) = 1 - \frac{a^2}{z^2} - \frac{ic}{z} \quad (z \in \mathbb{C} - \{0\})$$

and complex potential function

$$\Omega_{a,c}(z) = z + \frac{a^2}{z} - ic \operatorname{Log} z \quad (z \in \mathbb{C}_\pi)$$

has the following properties:

- (a) $\lim_{z \rightarrow \infty} q_{a,c}(z) = 1$
- (b) $\partial K_a = C_a$ is made up of streamlines for the flow
- (c) for any simple-closed contour Γ surrounding K_a ,
 - (i) $\mathcal{C}_\Gamma = \operatorname{Re} \int_\Gamma \overline{q_{a,c}}(z) dz = 2\pi c$
 - (ii) $\mathcal{F}_\Gamma = \operatorname{Im} \int_\Gamma \overline{q_{a,c}}(z) dz = 0$.

3. An **obstacle** is a compact, connected set K in \mathbb{C} , which is not a single point, such that $\mathbb{C} - K$ is also connected.

4. **Obstacle Problem** Given an obstacle K and a real number c , we seek a velocity function q for an ideal flow with flow region $\mathcal{R} = \mathbb{C} - K$ satisfying the following properties.

- (a) $\lim_{z \rightarrow \infty} q(z) = 1$.
- (b) There is a complex potential function Ω for q on either \mathcal{R} or $\mathcal{R} - \Sigma$, where Σ is a simple smooth path in \mathcal{R} joining a point of K to ∞ , and a real constant k such that

$$\lim_{z \rightarrow \alpha} \operatorname{Im} \Omega(z) = k, \quad \text{for each } \alpha \in \partial K.$$

- (c) For any simple-closed contour Γ surrounding K ,

$$\mathcal{C}_\Gamma = 2\pi c.$$

5. The quantity $2\pi c$ in the Obstacle Problem is called the **circulation around the obstacle K** .
6. The ideal flow with velocity function $q_{a,c}$ solves the Obstacle Problem for $K = K_a$ with circulation $2\pi c$ around K .

7. **Flow Mapping Theorem** Let K be an obstacle, and let f be a one-to-one conformal mapping from $\mathbb{C} - K$ onto $\mathbb{C} - K_a$, where $a > 0$, such that the Laurent series about 0 for f on $\{z : |z| > R\}$ has the form

$$f(z) = z + a_0 + \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \cdots, \quad \text{for } |z| > R,$$

where $R > 0$ and $a_0, a_{-1}, a_{-2}, \dots \in \mathbb{C}$. Then the velocity function

$$q(z) = q_{a,c}(f(z))\overline{f'(z)} \quad (z \in \mathbb{C} - K)$$

is the unique solution to the Obstacle Problem for K with circulation $2\pi c$ around K , and a corresponding complex potential function is

$$\Omega(z) = \Omega_{a,c}(f(z)).$$

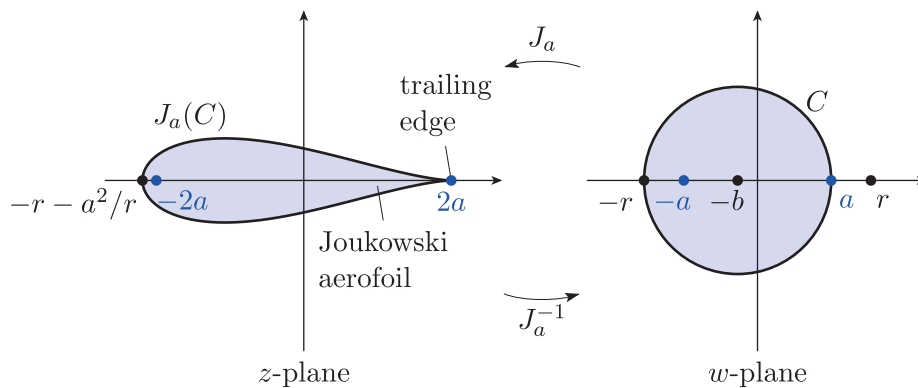
8. Let $J_\alpha(w) = w + \alpha^2/w$, where $\alpha \neq 0$, and let J_α^{-1} be the inverse function of the restriction of J_α to $\mathbb{C} - K_{|\alpha|}$. Then, for $z \in \mathbb{C} - L(-2\alpha, 2\alpha)$,

- (a) $J_\alpha^{-1}(z) + \alpha^2/J_\alpha^{-1}(z) = z$
 (b) $J_\alpha^{-1}(z) - \alpha^2/J_\alpha^{-1}(z) = z\sqrt{1 - 4\alpha^2/z^2}$
 (c) $(J_\alpha^{-1})'(z) = \frac{1}{1 - \alpha^2/(J_\alpha^{-1}(z))^2} = \frac{J_\alpha^{-1}(z)}{z\sqrt{1 - 4\alpha^2/z^2}}.$

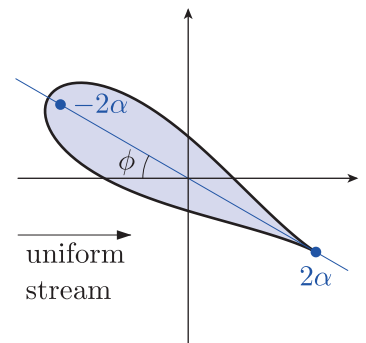
Section 5 Flow past an aerofoil

1. A point at which a function f has zero derivative is called a **critical point** of f .
2. A **Joukowski aerofoil** is an obstacle that has boundary $J_a(C)$, for $a > 0$ (possibly after an appropriate translation or rotation), where C is a circle that passes through one of the critical points $w = a$ of J_a and surrounds the other critical point $w = -a$.

The point $z = 2a$ is called the **trailing edge** of the aerofoil.



3. A symmetric aerofoil in a uniform stream in the direction of the positive x -axis has **angle of attack** ϕ , where ϕ is the angle from the line of symmetry of the aerofoil to the negative x -axis.



Unit D2 The Mandelbrot set

Section 1 Iteration of analytic functions

1. A sequence (z_n) defined by a recurrence relation of the form

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

where f is a function, is called an **iteration sequence** with **initial term** z_0 .

2. Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be a function. The **n th iterate** of f is the function f^n obtained by applying f exactly n times:

$$f^n = f \circ f \circ \dots \circ f, \quad \text{where } n = 1, 2, \dots$$

Also, f^0 denotes the identity function $f^0(z) = z$.

3. A **fixed point** of a function f is a point α for which $f(\alpha) = \alpha$.
4. The equation $f(z) = z$ is called the **fixed point equation**.
5. **Theorem** Let α be a fixed point of an analytic function f , and suppose that $|f'(\alpha)| < 1$. Then there exists $r > 0$ such that

$$\lim_{n \rightarrow \infty} f^n(z_0) = \alpha, \quad \text{for } |z_0 - \alpha| < r.$$

6. A fixed point α of an analytic function f is
 - **attracting** if $|f'(\alpha)| < 1$
 - **repelling** if $|f'(\alpha)| > 1$
 - **indifferent** if $|f'(\alpha)| = 1$
 - **super-attracting** if $f'(\alpha) = 0$.
7. Let α be an attracting fixed point of an analytic function f . Then the **basin of attraction** of α under f is the set

$$\{z : f^n(z) \rightarrow \alpha \text{ as } n \rightarrow \infty\}.$$

8. The functions f and g are **conjugate** to each other if

$$g = h \circ f \circ h^{-1},$$

for some one-to-one function h called the **conjugating function**.

Let (z_n) be the sequence defined by

$$z_{n+1} = f(z_n), \quad n = 0, 1, 2, \dots,$$

for some initial term z_0 , and let $w_n = h(z_n)$, for $n = 0, 1, 2, \dots$. Then the sequence (w_n) satisfies

$$w_{n+1} = g(w_n), \quad \text{for } n = 0, 1, 2, \dots,$$

and (z_n) and (w_n) are called **conjugate iteration sequences**.

Section 2 Iterating complex quadratics

1. **Theorem** The iteration sequence

$$z_{n+1} = az_n^2 + bz_n + c, \quad n = 0, 1, 2, \dots,$$

where $a \neq 0$, is conjugate to the iteration sequence

$$w_{n+1} = w_n^2 + d, \quad n = 0, 1, 2, \dots,$$

where $d = ac + \frac{1}{2}b - \frac{1}{4}b^2$. The conjugating function is

$$h(z) = az + \frac{1}{2}b.$$

2. Functions of the form

$$P_c(z) = z^2 + c, \quad \text{where } c \in \mathbb{C},$$

are called **basic quadratic functions**.

3. The fixed points of P_c are $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$.

4. **Theorem** Let $r_c = \frac{1}{2} + \sqrt{\frac{1}{4} + |c|}$. Then, for $|z_0| > r_c$,

$$|P_c^n(z_0)|, \quad n = 0, 1, 2, \dots,$$

is an increasing sequence, and

$$P_c^n(z_0) \rightarrow \infty \text{ as } n \rightarrow \infty.$$

5. For $c \in \mathbb{C}$, the **escape set** of P_c is

$$E_c = \{z : P_c^n(z) \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

The complement of E_c is denoted by K_c and is called the **keep set**.

6. A set A is **completely invariant** under a function f if

$$z \in A \iff f(z) \in A.$$

7. **Theorem** For each $c \in \mathbb{C}$, the escape set E_c and the keep set K_c have the following properties:

- (a) $E_c \supseteq \{z : |z| > r_c\}$ and $K_c \subseteq \{z : |z| \leq r_c\}$
- (b) E_c is open and K_c is closed
- (c) $E_c \neq \mathbb{C}$ and $K_c \neq \emptyset$
- (d) E_c and K_c are each completely invariant under P_c
- (e) E_c and K_c are each symmetric under rotation by π about 0
- (f) E_c is (pathwise) connected and K_c has no holes in it.

8. K_c is a compact set.

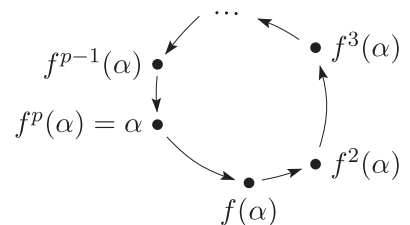
9. The point α is a **periodic point**, with **period** p , of a function f if

$$f^p(\alpha) = \alpha, \text{ but } f^k(\alpha) \neq \alpha, \text{ for } k = 1, 2, \dots, p-1.$$

The p points

$$\alpha, f(\alpha), f^2(\alpha), \dots, f^{p-1}(\alpha)$$

then form a **cycle of period** p , or a **p -cycle**, of f .



10. Any periodic point α of P_c lies in the keep set K_c .
11. **Theorem** Let $\alpha, f(\alpha), f^2(\alpha), \dots, f^{p-1}(\alpha)$ form a p -cycle of an analytic function f .
 - (a) Then the derivative of f^p at α satisfies

$$(f^p)'(\alpha) = f'(\alpha) \times f'(f(\alpha)) \times f'(f^2(\alpha)) \times \dots \times f'(f^{p-1}(\alpha)),$$
 and hence the derivative of f^p takes the same value at each point of the p -cycle; that is,

$$(f^p)'(\alpha) = (f^p)'(f(\alpha)) = (f^p)'(f^2(\alpha)) = \dots = (f^p)'(f^{p-1}(\alpha)).$$
 - (b) Let $g = h \circ f \circ h^{-1}$, where h is a one-to-one analytic function, and let $\beta = h(\alpha)$. Then $\beta, g(\beta), g^2(\beta), \dots, g^{p-1}(\beta)$ is a p -cycle of g , and

$$(g^p)'(\beta) = (f^p)'(\alpha).$$
12. Let α be a periodic point with period p of an analytic function f . Then the number $(f^p)'(\alpha)$ is called the **multiplier** of the corresponding p -cycle.
13. Let α be a periodic point with period p of an analytic function f . Then α and the corresponding p -cycle are
 - **attracting** if $|(f^p)'(\alpha)| < 1$
 - **repelling** if $|(f^p)'(\alpha)| > 1$
 - **indifferent** if $|(f^p)'(\alpha)| = 1$
 - **super-attracting** if $(f^p)'(\alpha) = 0$.
14. **Theorem** Let α be a periodic point of the function P_c .
 - (a) If α is attracting, then α is an interior point of K_c .
 - (b) If α is repelling, then α is a boundary point of K_c .

Section 3 Graphical iteration

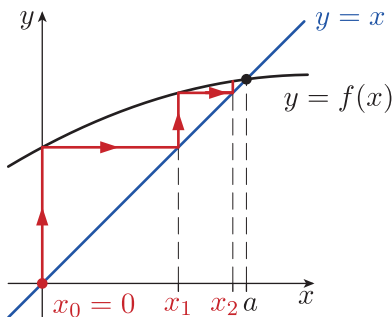
1. **Graphical iteration** with a real function f is the process of constructing the sequence (x_n) , where

$$x_{n+1} = f(x_n), \quad n = 0, 1, 2, \dots,$$

for a given point $x_0 \in \mathbb{R}$, by drawing alternately vertical and horizontal lines joining the points

$$\begin{aligned} &(x_0, 0) \text{ to } (x_0, x_1) = (x_0, f(x_0)) \\ &(x_0, x_1) \text{ to } (x_1, x_1) \\ &(x_1, x_1) \text{ to } (x_1, x_2) = (x_1, f(x_1)) \\ &(x_1, x_2) \text{ to } (x_2, x_2) \\ &(x_2, x_2) \text{ to } (x_2, x_3) = (x_2, f(x_2)), \\ &\text{and so on,} \end{aligned}$$

using the graphs of $y = f(x)$ and $y = x$.



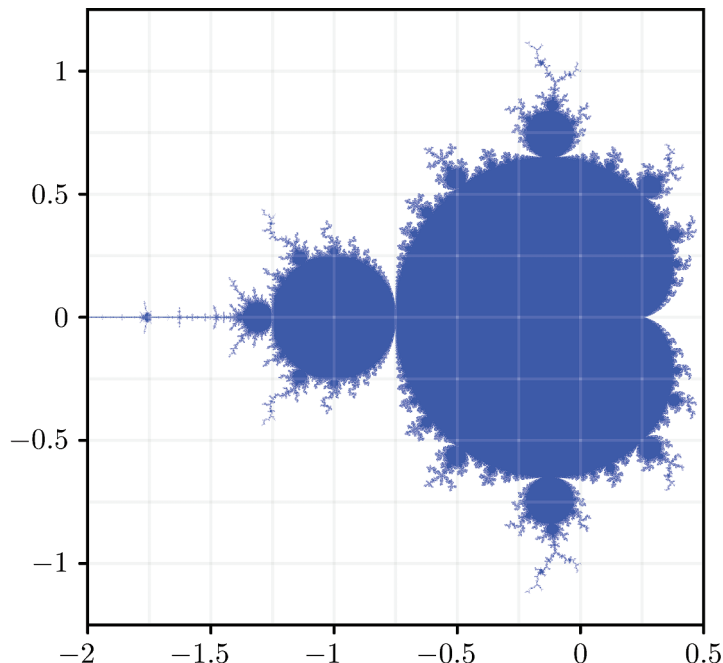
2. For $c \in \mathbb{R}$, the function P_c has
 - (a) no real fixed points if $c > \frac{1}{4}$
 - (b) a single fixed point $\frac{1}{2}$, if $c = \frac{1}{4}$
 - (c) two real fixed points $\frac{1}{2} \pm \sqrt{\frac{1}{4} - c}$, if $c < \frac{1}{4}$.
3. **Theorem** If $c > \frac{1}{4}$, then $K_c \cap \mathbb{R} = \emptyset$.
4. For $c \leq \frac{1}{4}$, I_c denotes the closed interval

$$I_c = \left[-\frac{1}{2} - \sqrt{\frac{1}{4} - c}, \frac{1}{2} + \sqrt{\frac{1}{4} - c} \right].$$
5. **Theorem** If $-2 \leq c \leq \frac{1}{4}$, then $K_c \cap \mathbb{R} = I_c$.
6. **Theorem** If $c < -2$, then the set $K_c \cap \mathbb{R}$ consists of the closed interval I_c with a sequence of disjoint, non-empty, open subintervals of I_c removed. In particular, $0 \notin K_c$.

Section 4 The Mandelbrot set

1. A set A is **disconnected** if there are disjoint open sets G_1 and G_2 such that

$$A \cap G_1 \neq \emptyset, \quad A \cap G_2 \neq \emptyset \quad \text{and} \quad A \subseteq G_1 \cup G_2.$$
 A set A is **connected** if it is not disconnected.
2. The set K_c is disconnected when $c > \frac{1}{4}$ and when $c < -2$.
3. **Theorem** Any pathwise connected set is connected.
However, a connected set need not be pathwise connected.
4. The **Mandelbrot set** is the set M of complex numbers c such that K_c is connected.



5. **Fatou–Julia Theorem** For any $c \in \mathbb{C}$,

$$K_c \text{ is connected} \iff 0 \in K_c.$$

6. **Theorem** The Mandelbrot set M can be specified as

$$M = \{c : |P_c^n(0)| \leq 2, \text{ for } n = 1, 2, \dots\}.$$

7. The Mandelbrot set M

- (a) is a compact subset of $\{c : |c| \leq 2\}$
- (b) is symmetric under reflection in the real axis
- (c) meets the real axis in the interval $[-2, \frac{1}{4}]$
- (d) has no holes in it; that is, $\mathbb{C} - M$ is connected.

8. **Theorem** The Mandelbrot set is connected.

9. Suppose that $c \neq -\frac{3}{4}$. Then P_c has a single 2-cycle α_1, α_2 , where

$$\alpha_1 = -\frac{1}{2} + \sqrt{-\frac{3}{4} - c}, \quad \alpha_2 = -\frac{1}{2} - \sqrt{-\frac{3}{4} - c},$$

with multiplier

$$(P_c^2)'(\alpha_1) = 4\alpha_1\alpha_2 = 4(c + 1).$$

10. **Theorem** If the function P_c has an attracting cycle, then $c \in M$.

11. **Theorem**

- (a) The function P_c has an attracting fixed point if and only if c satisfies

$$(8|c|^2 - \frac{3}{2})^2 + 8 \operatorname{Re} c < 3.$$

- (b) The function P_c has an attracting 2-cycle if and only if c satisfies

$$|c + 1| < \frac{1}{4}.$$

12. A **periodic region** is a maximal region \mathcal{R} such that, for some positive integer p , the function P_c has an attracting p -cycle, for all $c \in \mathcal{R}$.

13. **Theorem** The function P_c has a super-attracting p -cycle if and only if

$$P_c^p(0) = 0, \quad \text{but } P_c^k(0) \neq 0, \text{ for } k = 1, 2, \dots, p-1.$$

14. The number λ is a **primitive n th root of unity** if λ is a root of unity and if n is the smallest positive integer for which $\lambda^n = 1$.

15. **Theorem** Suppose that the function P_{c_0} , where $c_0 \in \mathbb{C}$, has a p -cycle whose multiplier λ is a root of unity.

- (a) **Saddle-node bifurcation at c_0** If $\lambda = 1$, then c_0 is the cusp of a cardioid-shaped periodic region \mathcal{R} , such that

$$P_c \text{ has an attracting } p\text{-cycle, for } c \in \mathcal{R}.$$

- (b) **Period-multiplying bifurcation at c_0** If λ is a primitive n th root of unity, for $n > 1$, then there are two periodic regions \mathcal{R}_1 and \mathcal{R}_2 whose boundaries meet at c_0 such that

$$P_c \text{ has an attracting } \begin{cases} p\text{-cycle,} & \text{for } c \in \mathcal{R}_1, \\ np\text{-cycle,} & \text{for } c \in \mathcal{R}_2. \end{cases}$$

Index

- \overline{A} 69
- $\binom{\alpha}{n}$ 52
- $\operatorname{Arg} z$ 16
- $\operatorname{Arg}_{\phi}(z)$ 64
- \mathbb{C} 13
- $\widehat{\mathbb{C}}$ 73
- \mathbb{C}_{ϕ} 64
- C_a 86
- $\cos z$ 27
- $\operatorname{cosec} z$ 27
- $\operatorname{cosech} z$ 27
- $\cosh z$ 27
- $\cot z$ 27
- $\coth z$ 27
- $\frac{df}{dz}$ 36
- E_c 89
- \emptyset 5
- $\exp z$ 25
- $\operatorname{ext} A$ 35
- e^z 25
- f^{-1} 22
- f' 36
- f'' 36
- $f^{(n)}$ 36
- f^n 88
- $f(r) \rightarrow \alpha$ as $r \rightarrow \infty$ 60
- $f(z) \rightarrow \beta$ as $z \rightarrow \alpha$ 32
- $f(z) \rightarrow \beta$ as $z \rightarrow \infty$ 73
- $f(z) \rightarrow \infty$ as $z \rightarrow \alpha$ 55
- $\tilde{\Gamma}$ 42
- $\Gamma(z)$ 71
- i 13
- I_c 91
- $\operatorname{Im} z$ 13
- $\inf A$ 6
- ∞ 73
- $\operatorname{int} A$ 35
- $\int_{\Gamma} f$ 41
- $J(z)$ 85
- $J_{\alpha}(z)$ 85
- K_a 86
- K_c 89
- $L(\Gamma)$ 43
- $L(-2\alpha, 2\alpha)$ 85
- $\lim_{n \rightarrow \infty} z_n$ 29
- $\lim_{r \rightarrow \infty} f(r)$ 60
- $\lim_{z \rightarrow \alpha} f(z)$ 32
- $\operatorname{Log} z$ 28
- $\operatorname{Log}_{\phi}(z)$ 64
- $\max A, \min A$ 6
- \mathbb{N} 6
- $\binom{n}{k}$ 14
- $\Omega(z)$ 82
- $\Omega_{a,c}(z)$ 86
- ∂A 35
- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ 38
- $P_c(z)$ 89
- $\Phi(z)$ 82
- $\Psi(z)$ 82
- \mathbb{Q} 6
- $q_{a,c}(z)$ 86
- $q_N(s)$ 81
- $q_{\theta}(z)$ 80
- $q_T(s)$ 81
- \mathbb{R} 6
- $\operatorname{Res}(f, \alpha)$ 58
- $\operatorname{Re} z$ 13
- \mathbb{S} 73
- $\sec z$ 27
- $\operatorname{sech} z$ 27
- $\sin z$ 27
- $\sinh z$ 27
- $\sup A$ 6
- $\tan z$ 27
- $\tanh z$ 27
- $\operatorname{Wnd}(\Gamma, 0)$ 66
- $\operatorname{Wnd}(\Gamma, \alpha)$ 66
- \mathbb{Z} 6
- z^{-1} 13
- z^{α} 28
- \bar{z} 13
- $|z|$ 15
- (z_n) 29
- $z_n \rightarrow \alpha$ as $n \rightarrow \infty$ 29
- $z_n \rightarrow \infty$ as $n \rightarrow \infty$ 30
- $\zeta(z)$ 71

- Absolute Convergence Test 48
- absolute value 15
- absolutely convergent 48
- agree on a set 54
- analytic 36
 - at a point 36
 - on a region 36
- analytic continuation 65
 - direct 64
 - indirect 65
- analytic extension 55
- analytic part of an extended power series 56
- analyticity of derivatives 45
- angle of a lune 77
- angle of attack 87
- annulus 19
 - closed 19
 - open 19
- annulus of convergence 56
- anticlockwise vortex 81
- Apollonian form 75
- arc of a circle 24
- argument 16
 - principal 16
- Argument Principle 67
- arithmetic in \mathbb{C} 14
- associativity 74

- basic continuous functions 32
- basic null sequences 29
- basic quadratic functions 89
- basic regions 34
- basic Taylor series 52
- basin of attraction 88
- bifurcation
 - period-multiplying 92
 - saddle-node 92
- binomial coefficients
 - in the Binomial Theorem 14
 - of binomial series 52
- binomial series 52
- Binomial Theorem 15
- boundary
 - of a set 35
 - of a set in $\widehat{\mathbb{C}}$ 76
- boundary point
 - in $\widehat{\mathbb{C}}$ 76
 - of a set 35
- Boundary Uniqueness Theorem 69
- bounded function 35
- bounded sequence 30
- bounded set 34
- Boundedness Theorem 35

- Cartesian form 13
- Cartesian grid 25
- Casorati–Weierstrass Theorem 58
- Cauchy’s First Derivative Formula 45
- Cauchy’s Integral Formula 45
- Cauchy’s n th Derivative Formula 45
- Cauchy’s Residue Theorem 59
- Cauchy’s Theorem 44
- Cauchy–Riemann Converse Theorem 39
- Cauchy–Riemann equations 38
- Cauchy–Riemann Theorem 38
- chain of functions 65
 - closed 65
- Chain Rule 39
- circle 24
 - unit 19
- circulation 81
 - around an obstacle 86
- Circulation and Flux Contour Integral 81
- clockwise vortex 81
- closed annulus 19
- closed chain of functions 65
- Closed Contour Theorem 43
- closed disc 19
- closed half-plane 18
- closed path or contour 43
- closed set 34
- closure of a set 69
- codomain 21
- Combination Rules
 - for closed sets 34
 - for continuous functions 32
 - for contour integrals 42
 - for differentiation 37
 - for limits of functions 33, 60
 - for open sets 33
 - for power series 52
 - for sequences 30
 - for series 48
- common ratio 47
- compact set 34
- Comparison Test 48

- complement of a set 20
- completely invariant set 89
- complex conjugate 13
- complex function 21
- complex numbers 13
 - arithmetic 13
 - conjugate 13
 - equality 13, 16
 - reciprocal 13
 - roots 17
- complex plane 15
 - extended 73
- complex potential function 82
- complex sequence 29
- complex series 47
- composite function 22
- Composition Rule
 - for continuous functions 32
 - for power series 53
- conformal 40
 - at a point 40
 - on a set 40
- conformal mapping 40, 72
- conjugate functions 88
- conjugate iteration sequences 88
- conjugate velocity function 80
- connected set 91
 - pathwise 34
- continuous argument function 66
- continuous function 31
 - ε - δ definition 31
 - basic 32
 - sequential definition 31
- contour 41
 - closed 43
 - final point 41
 - initial point 41
 - integral 41
 - length 43
 - reverse 42
 - simple 44
 - simple-closed 44
- Contour Independence Theorem 42, 44
- convention
 - domain and codomain 21
 - Möbius transformation 74
 - orientation of a contour 44
 - sketching 20
- convergence of a sequence of functions
 - pointwise 69
 - uniform 69
- convergent
 - extended power series 56
 - sequence 29
 - series 47
- convex set 34
- Cover-up Rule 59
- critical point 87
- cut plane 19, 64
- cycle 89
- De Moivre's Theorem 17, 26
- degenerate streamline 80
- degree of a polynomial 22
- derivative 36
 - geometric interpretation of 37
 - higher-order 36
 - standard 11, 39
- difference of sets 20
- differentiable 36
 - at a point 36
 - on a set 36
- Differentiation Rule for Power Series 50
- direct analytic continuation 64
- disc 19
 - closed 19
 - of convergence 50
 - open 19
 - punctured 19
- disconnected set 91
- discontinuous function 31
- divergent
 - sequence 30
 - series 47
- domain 21
- dominant term 67
- ellipse 24
- empty set 5
- endpoints of a path 23
- entire function 36
- escape set 89
- essential singularity 55
- Estimation Theorem 43
- Euler's Equation 26

- Euler's Identity 25
- even function 51
- even subsequence 30
- exponential function 25
 - common values 26
 - geometric nature 26
- exponential identities 26
- extended complex plane 73
- extended line 73
- extended power series 56
 - analytic part 56
 - singular part 56
 - sum 56
 - sum function 56
- exterior of a set 35
- exterior point 35
- Extreme Value Theorem 34

- Fatou–Julia Theorem 92
- final point
 - of a contour 41
 - of a path 23
- First Subsequence Rule 30
- fixed point 88
 - attracting 88
 - equation 88
 - indifferent 88
 - repelling 88
 - super-attracting 88
- fixed point of a Möbius transformation 74
- flow 80
 - around a vortex 84
 - continuous 80
 - due to a doublet 84
 - due to a doublet in a uniform stream 84
 - ideal 81
 - near a stagnation point 83
 - steady 80
 - two-dimensional 80
 - uniform 80, 83
 - with a source or sink 83
- flow line 80
- Flow Mapping Theorem 87
- fluid flow 80
- flux 81
- function
 - analytic 36
 - bounded 35
 - complex 21
 - complex potential 82
 - composite 22
 - conformal 40
 - conjugate 88
 - continuous 31
 - critical point 87
 - derivative 36
 - differentiable 36
 - discontinuous 31
 - entire 36
 - even 51
 - exponential 25
 - fixed point 88
 - flow velocity 80
 - hyperbolic 27
 - image set 21
 - imaginary part 22
 - inverse 22
 - Joukowski 85
 - limit 32, 60
 - linear 72
 - logarithm 64
 - n -to-one 68
 - n th iterate 88
 - odd 51
 - one-to-one 22
 - onto 21
 - polynomial 22
 - primitive 42
 - principal logarithm 28
 - principal power 28
 - rational 22
 - real 21
 - real part 22
 - real-valued 21
 - reciprocal 72
 - residue 58
 - restriction 22
 - stream 82
 - tends to infinity 55
 - trigonometric 27
 - unbounded 35
 - zeros 21
- functional equation for the gamma function 71
- Fundamental Theorem of Algebra 45
- Fundamental Theorem of Calculus 42

- g/h Rule 59
- gamma function 71
 - functional equation 71
- Gaussian integral 71
- generalised circle 73
- generalised open disc 76
- geometric series 47
- Geometric Series Identity 15
- graphical iteration 90
- graphs of standard functions 10–11
- Greek alphabet 4
- grid
 - Cartesian 25
 - orthogonal 40
 - polar 25
- grid path 43
- Grid Path Theorem 43
- group properties 74
- half-line 19
- half-plane 18
- harmonic series 47
- higher-order derivative 36
- hyperbola 25
- hyperbolic functions 27
- hyperbolic identities 9, 28
- ideal flow 81
- image
 - of a point 21
 - of a set 22
- image path 23
- image set 21
- imaginary axis 15
- imaginary part of a complex number 13
- imaginary part of a function 22
- improper integral 61, 64
- indirect analytic continuation 65
- inequalities 7, 18
- infinite series 47
- initial point
 - of a contour 41
 - of a path 23
- initial term 88
- inside of a simple-closed path 44
- integral
 - along a contour 41
 - along a path 41
 - improper 61, 64
 - standard 12
- Integration by Parts 42
- Integration Rule for Power Series 50
- interior of a set 35
- interior point 35
- intersection of sets 20
- inverse function 22
 - of a Möbius transformation 74
- Inverse Function Rule 39, 68
- inverse points 75
- inverse points method 76
- isolated singularity 55
- isolated zero 54
- iteration sequence 88
- Jordan Curve Theorem 44
- Jordan's Lemma 62
- Joukowski aerofoil 87
- Joukowski function 85
- keep set 89
- Laurent series 57
 - for \cot and cosec 63
- Laurent's Theorem 57
- left half-plane 18
- length
 - of a contour 43
 - of a path 43
- limit 32
 - ε - δ definition 33
 - of a function 32, 60
 - sequential definition 32
- limit function 69
- limit point 32
- line 24
- line segment 24
- Linear Approximation Theorem 36
- linear function 72
- Liouville's Theorem 45
- Local Mapping Theorem 68
- local maximum 69
- Local Maximum Principle 69
- locally circulation-free 81
- locally flux-free 81
- logarithm 64
 - principal 28

- logarithmic derivative 67
- logarithmic identities 8, 28
- lower half-plane 18
- lune 77
- Mandelbrot set 91
- mapping 21
- Maximum Principle 69
- Minimum Principle 69
- Möbius transformation 74
 - explicit formula 75
 - group properties 74
 - implicit formula 75
 - inverse function 74
- modulus 7, 15
- Morera's Theorem 46
- M -test 70
- multiple of a function 21
- Multiple Rule
 - for continuous functions 32
 - for contour integrals 42
 - for differentiation 37
 - for limits of functions 33, 60
 - for power series 52
 - for sequences 30
 - for series 48
- multiplier 90
- non-negative real axis 65
- Non-null Test 47
- normal component
 - of a velocity function on a path 81
- North Pole 73
- n th derivative 36
- n th iterate of a function 88
- n th partial sum 47
- n th roots
 - of a complex number 17
 - of unity 18
- n th term of a series 47
- n th term of a sequence 29
- n -to-one 68
- null sequence 29
- obstacle 86
- Obstacle Problem 86
- odd function 51
- odd subsequence 30
- one-to-one function 22
- onto function 21
- open annulus 19
- open disc 19
- open disc centred at ∞ 76
- open half-plane 18
- Open Mapping Theorem 68
- open sector 19
- open set 33
- order
 - of a pole 55
 - of a zero 54
- orthogonal 40
- outside of a simple-closed path 44
- parabola 25
- parametric equations 23
- parametrisation 23
 - smooth 40
 - standard 24
 - unit-speed 80
- partial derivative 38
- partial fraction expansion 46
- partial sum 47
- partial sum functions 70
- path 23
 - closed 43
 - direction 23
 - grid 43
 - image 23
 - integral 41
 - length 43
 - reverse 42
 - simple 44
 - simple-closed 44
 - smooth 40
- pathwise connected set 34
- p -cycle 89
- period-multiplying bifurcation 92
- periodic point 89
 - attracting 90
 - indifferent 90
 - repelling 90
 - super-attracting 90
- periodic region 92
- plane
 - cut 19, 64
 - punctured 19

- point
 - exterior 35
 - final 23
 - initial 23
 - interior 35
 - limit 32
- point at ∞ 73
- pointwise convergence
 - of a sequence of functions 69
 - of a series of functions 70
- polar coordinates 16
- polar form 16, 26
- polar grid 25
- pole
 - of order k 55
 - simple 55
- polynomial function 22
- power series 49
 - extended 56
 - sum function 49
- primitive n th root of unity 92
- primitive of a function 42
- Primitive Theorem 46
- principal n th root 18
- principal argument 16
- principal logarithm 28
 - geometric nature 28
- principal logarithm function 28
- principal power function 28
- Principle of Mathematical Induction 7
- product of functions 21
- Product Rule
 - for continuous functions 32
 - for differentiation 37
 - for inequalities 7
 - for limits of functions 33, 60
 - for power series 53
 - for sequences 30
- punctured disc 19
- punctured plane 19
- quadrants 15
- quadratic formula 18
- quotient
 - of complex numbers 13
 - of functions 21
- Quotient Rule
 - for continuous functions 32
 - for differentiation 37
 - for limits of functions 33, 60
 - for sequences 30
- radius of convergence 49
- Radius of Convergence Formula 50
- Radius of Convergence Theorem 49
- Ratio Test 48
- rational function 22
- ray 19
- real axis 15
- real function 21
- real part of a complex number 13
- real part of a function 22
- real series 47
- real-valued function 21
- reciprocal function 72
- Reciprocal Rule
 - for differentiation 37
 - for functions 55
 - for sequences 30
- region 34
 - basic 34
 - periodic 92
 - simply connected 44
- removable singularity 55
- residue 58
- Residue Theorem 59
- restriction of a function 22
- Restriction Rule
 - for continuous functions 32
 - for differentiation 39
- reverse contour 42
- Reverse Contour Theorem 42
- reverse path 42
- Riemann sphere 73
- right half-plane 18
- roots
 - of a complex number 17
 - square 17
- rotation 72
- Rouché's Theorem 67
- Round-the-Pole Lemma 61
- saddle-node bifurcation 92
- scaling 72

- Schwarz's Lemma 69
- Second Subsequence Rule 30
- sector 19
 - open 19
- sequence 29
 - bounded 30
 - complex 29
 - constant 29
 - convergent 29
 - divergent 30
 - iteration 88
 - n th term 29
 - null 29
- series 47
 - absolutely convergent 48
 - binomial 52
 - complex 47
 - convergent 47
 - divergent 47
 - geometric 47
 - harmonic 47
 - infinite 47
 - Laurent 57
 - power 49
 - real 47
 - Taylor 51
- set
 - bounded 34
 - closed 34
 - closure 69
 - compact 34
 - complement 20
 - connected 91
 - convex 34
 - disconnected 91
 - empty 5
 - Mandelbrot 91
 - open 33
 - unbounded 34
- set notation 5
- sets
 - difference 20
 - intersection 20
 - union 20
- Shrinking Contour Theorem 45
- simple contour 44
- simple path 44
- simple pole 55
- simple zero 54
- simple-closed contour 44
- simple-closed path 44
- simply connected region 44
- \sin^{-1} function 68, 79
- singular part of an extended power series 56
- singularity 55, 58
 - essential 55
 - isolated 55
 - removable 55
- sink 81
- sketching conventions 20
- smooth parametrisation 40
- smooth path 40
- source 81
- square root of a complex number 17
- Squeeze Rule for sequences 29
- stagnation point 80
- standard conformal mappings 78
- standard derivatives 11, 39
- standard integrals 12
- standard parametrisations 24
- standard primitives 12
- stereographic projection 73
- strategy
 - for determining an image path 23
 - for determining principal arguments 16
 - for determining whether a function is continuous 31
 - for evaluating contour integrals 46
 - for evaluating real trigonometric integrals 60
 - for finding n th roots 18
 - for finding an equation for the image of a path under the reciprocal function 72
 - for inverting a Taylor series 68
 - for mapping lunes 77
 - for non-differentiability 38
 - for obtaining a quotient 13
 - for proving that a limit does not exist 33
 - for proving that an inverse function exists 22
 - for proving uniform convergence 70
- stream function 82
- streamline 80
 - degenerate 80

- strength
 - of a sink 81
 - of a source 81
 - of a vortex 81
- subsequence 30
 - even 30
 - odd 30
- Subsequence Rules 30
- substitution method 75
- Substitution Rule for power series 53
- sum
 - of a series 47
 - of an extended power series 56
- sum function
 - of a power series 49
 - of a series of functions 70
 - of an extended power series 56
- sum of functions 21
- Sum Rule
 - for continuous functions 32
 - for contour integrals 42
 - for differentiation 37
 - for inequalities 7
 - for limits of functions 33, 60
 - for power series 52
 - for sequences 30
 - for series 48
- \tan^{-1} function 68, 79
- tangent vector 40
- tangential component
 - of a velocity function on a path 81
- Taylor coefficient 51
- Taylor series 51
- Taylor's Theorem 51
- three-point trick 75
- trailing edge 87
- transformation 21
- Transitive Rule 7
- translation 72
- Triangle Inequality 20
 - for Series 48
- trigonometric functions 27
- trigonometric identities 9, 27
- trigonometric values commonly used 9
- unbounded function 35
- unbounded set 34
- uniform convergence
 - of a sequence of functions 69
 - of a series of functions 70
- uniform flow 80, 83
- uniform stream 80
- union of sets 20
- Uniqueness Theorem 54
- unit circle 19
- unit-speed parametrisation 80
- upper half-plane 18
- velocity function 80
- vertices of a lune 77
- vortex 81
 - anticlockwise 81
 - clockwise 81
- Weierstrass' M -test 70
- Weierstrass' Theorem 71
- winding number
 - around a point 66
 - around the origin 66
- zero
 - isolated 54
 - of order k 54
 - simple 54
- Zero Derivative Theorem 43
- zeros 21
 - of \cos 27
 - of \cosh 28
 - of \sin 27
 - of \sinh 28
- zeta function 71
- z -plane 15

